

# Fundamental group of Galois covers of degree 5 surfaces

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## Abstract

Let  $X$  be a surface of degree 5, which is considered as a branch cover of  $\mathbb{CP}^2$  with respect to a generic projection. The surface has a natural Galois cover with Galois group  $S_n$ . In this paper, we compute the fundamental groups of Galois covers of degree 5 that degenerate to nice plane arrangements; each of them is a union of five planes such that no three planes meet in a line. As an application, we give a counter-example of a question of Liedtke [29, Question 3.4].

## 1 Introduction

In 1977, Gieseker [25] proved that the moduli space of surfaces of general type is a quasi-projective variety. Unlike the case for curves, it is not irreducible. Catanese and Manetti [17, 18, 30] proved some results about the structure and number of components of moduli spaces of general type surfaces. However, still not much is known about such moduli spaces. In [42], Teicher defined some new invariants of surfaces, stable on connected components of moduli space. The new invariants come from the cyclic structure of the fundamental group of the complement of a branch curve.

Let  $X$  be an algebraic surface of degree  $n$ ; one can consider it as a branched cover of the projective plane  $\mathbb{CP}^2$  with respect to a generic projection  $\pi : X \rightarrow \mathbb{CP}^2$ . The branch curve  $S$  is an irreducible cuspidal plane curve of even degree, namely,  $S$  admits only nodes and ordinary cusps as its singularities. Chisini's conjecture [21], which was confirmed by Kulikov [27, 28], states that: If  $S$  is the branch curve of a generic projection  $\pi : X \rightarrow \mathbb{CP}^2$ , then  $\pi$  is determined uniquely by  $S$ , except

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for the case when  $X$  is the Veronese surface  $V_2$  in  $\mathbb{CP}^5$ . Thus, one can reduce the classification of algebraic surfaces to that of cuspidal branch curves.

It is well known that the fundamental group of the complement of the branch curve  $\pi_1(\mathbb{CP}^2 - S)$  (we always denote the branch curve by  $S$ , and denote the group  $\pi_1(\mathbb{CP}^2 - S)$  by  $G$ , for convenience) does not change when the complex structure of  $X$  changes continuously. Thus, we can use such an invariant to distinguish the connected components of the corresponding moduli space of surfaces. In fact, all surfaces in the same component of the moduli space have the same homotopy type and, therefore, the same fundamental group of the complement of the branch curves.

In [32] and [33], Moishezon-Teicher showed that  $\pi_1(\mathbb{CP}^2 - S)$  is connected with  $\pi_1(X_{\text{Gal}})$ , where the surface  $X_{\text{Gal}}$  is the Galois cover of  $X$ . Thus, we can calculate fundamental groups of some surfaces of general type that usually may be very difficult to determine. Based on this idea, Moishezon-Teicher [33] constructed a series of simply connected algebraic surfaces of general type, with positive and zero indices, thereby disproving the Bogomolov Watershed conjecture: An algebraic surface of general type, with a positive index, has an infinite fundamental group.

In recent years, some works were done in the study of  $\pi_1(\mathbb{CP}^2 - S)$  and  $\pi_1(X_{\text{Gal}})$  – for Cayley’s singularities [3]; for different embeddings of  $\mathbb{CP}^1 \times \mathbb{CP}^1$  [33]; for the Veronese surfaces  $V_n$ ,  $n \geq 3$  [37, 38], and for  $V_2$  [44]; for the Hirzebruch surfaces  $F_1(a, b)$ ,  $F_2(2, 2)$  [4, 23]; for  $T \times T$  where  $T$  is a complex torus [11, 13]; for  $K3$  surfaces [1]; for  $\mathbb{CP}^1 \times T$  [5, 6, 7]; for  $\mathbb{CP}^1 \times C_g$  where  $C_g$  is a curve of genus  $g$  [24]; and for certain toric surfaces [10]. In [15], one can also find a description of computations of braid monodromy and certain quotients of  $\pi_1(\mathbb{CP}^2 - S)$ ; the motivation came from the theory of symplectic 4-manifolds. In [29], Liedtke computed a quotient of  $\pi_1(X_{\text{Gal}})$  that depends on  $\pi_1(X)$  and data from the generic projection only, thereby simplifying the computations of Moishezon, Teicher, and others. In [9], the authors computed the fundamental groups of Galois covers of surfaces of degree  $\leq 4$ .

In this paper, we study the fundamental groups of the complements of the branch curves and of the Galois covers of degree 5 surfaces with nice planar degenerations (see Theorem 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9). As an application, we give a counter-example of the main unsolved question in [29], which asks: Is  $C^{\text{aff}}$  (see Section 4 for the definition) trivial for every generic projection of degree  $n \geq 5$ ?

This paper is organized as follows: In Section 2, we explain the main methods and give a fundamental and necessary background that we use in this paper. Section 3 is a complete calculation of the fundamental groups of the Galois covers of all the surfaces of degrees 5 that degenerate to ‘nice’ planar arrangements, i.e., those in which no three planes meet in a line. We consider the degenerations, give the braid monodromy and the group  $G$ , and then compute the group  $\pi_1(X_{\text{Gal}})$ . The surfaces that we consider are: the Hirzebruch surface  $F_1(2, 1)$  (Subsection 3.1), a union of the surface  $\mathbb{CP}^1 \times \mathbb{CP}^1$  and a plane (Subsection 3.2), a union of the Veronese surface  $V_2$  and a plane (Subsection 3.3), two cases of a union of the Cayley surface and two planes (Subsection 3.4), a union

of the quartic 4-point surface with a plane (Subsection 3.5), the quintic 5-point surface (Subsection 3.6), a 4-point quintic degeneration (Subsection 3.7). In Section 4, we give a counter-example of the main unsolved question in [29]. In the Appendix, we analyze an example of degree 4 that is missed in [9].

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## 2 Method and scientific background

In this section, we describe the main methods and the fundamental background used in this paper. The computations of the fundamental groups  $\pi_1(\mathbb{CP}^2 - S)$  and  $\pi_1(X_{\text{Gal}})$  of surfaces  $X$  of degree 5, with at worst isolated singularities and with nice degenerations, are explained here.

We start with an algebraic surface  $X$  embedded in a projective space  $\mathbb{CP}^n$ . We project it onto the projective plane  $\mathbb{CP}^2$  to get its branch curve  $S$ . Because it is not easy to describe  $S$ , we use a method called degeneration. The definition follows.

**Definition 2.1.** *Let  $D$  be the unit disc, and let  $X, Y$  be algebraic surfaces (or more generally, algebraic varieties). Let  $p : Y \rightarrow \mathbb{CP}^n$  and  $p' : X \rightarrow \mathbb{CP}^n$  be projective embeddings. We say that  $p'$  is a projective degeneration of  $p$  if there exists a flat family  $\pi : V \rightarrow D$  and an embedding  $F : V \rightarrow D \times \mathbb{CP}^n$ , such that  $F$  composed with the first projection is  $\pi$ , and:*

- (a)  $\pi^{-1}(0) \simeq X$ ;
- (b) there is a  $t_0 \neq 0$  in  $D$  such that  $\pi^{-1}(t_0) \simeq Y$ ;
- (c) the family  $V - \pi^{-1}(0) \rightarrow D - 0$  is smooth;
- (d) restricted to  $\pi^{-1}(0)$ ,  $F = 0 \times p'$  under the identification of  $\pi^{-1}(0)$  with  $X$ ;
- (e) restricted to  $\pi^{-1}(t_0)$ ,  $F = t_0 \times p$  under the identification of  $\pi^{-1}(t_0)$  with  $Y$ .

We perform a sequence of projective degenerations  $X := X_r \rightsquigarrow X_{r-1} \rightsquigarrow \cdots X_{r-i} \rightsquigarrow X_{r-(i+1)} \rightsquigarrow \cdots \rightsquigarrow X_0$ , and we refer to each step along the way as a partial degeneration ( $r$  is the number of partial degenerations). The surface  $X_0$  is a total degeneration of  $X$  if it is a union of linear spaces of dimension  $n$ .

In [16, Sec. 12], there are examples of surfaces that can projectively degenerate to a union of planes such that no 3 planes meet in a line. In our paper we deal with degree 5 surfaces; this means  $n = 5$ . For more technical details see [12].

**Example 2.1.** Consider the surface  $\mathbb{CP}^1 \times \mathbb{CP}^1$ . Take  $\ell_1 = \mathbb{CP}^1 \times pt$  and  $\ell_2 = pt \times \mathbb{CP}^1$ . For  $a, b \in \mathbb{N}$ , consider the linear combination  $a\ell_1 + b\ell_2$ . We embed our surface into a projective space via the linear system  $|a\ell_1 + b\ell_2|$ . For this example, we take  $a = 1$  and  $b = 2$  as in Figure 1.

We first degenerate the surface into two “squares”, such that each square is homeomorphic to  $\mathbb{CP}^1 \times \mathbb{CP}^1$ . Then we degenerate each square into two planes to get a total degeneration, which is a union of planes.

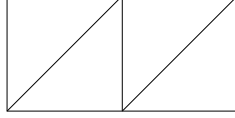


Figure 1: Degeneration of  $\mathbb{CP}^1 \times \mathbb{CP}^1$

Now, let us consider generic projections  $\pi^{(i)} : X_i \rightarrow \mathbb{CP}^2$  for  $0 \leq i \leq r$ . Let  $S_i$  be the branch curve of the generic projection for each  $i$ , and let  $S_{i-1}$  be a degeneration of  $S_i$  for  $1 \leq i \leq r$ . We regenerate  $S_0$  to get the regenerated branch curve  $S := S_r$ , using the regeneration Lemmas in [35].

In the following diagram (Figure 2), we illustrate the connections between the significant objects  $X$ ,  $X_0$ ,  $S$ , and  $S_0$ .

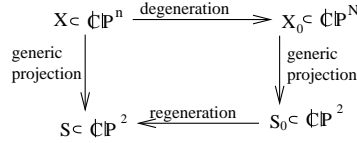


Figure 2: Diagram of degeneration-regeneration

Now we explain in general the regeneration process. Say that the degree of the degenerated branch curve  $S_0$  is  $m$ . Each of the  $m$  lines of  $S_0$  should be counted as a double line in the scheme-theoretic branch locus, as each arises from a nodal line. Another way to see this is to note that the regeneration of  $X_0$  induces a regeneration of  $S_0$  in such a way that each point on the typical fiber, say  $c$ , is replaced by two nearby points  $c, c'$ . This means that a line, say  $j$ , regenerates to two parallel lines or to a conic [12], and it is replaced by  $j$  and  $j'$ . The resulting branch curve  $S$  is of degree  $2m$ .

In full generality, the curve  $S_0$  may have  $k$ -point singularities for any value of  $k$ . A 1-point regenerates to a conic  $(j, j')$  with a branch point. A diagonal line in a 2-point regenerates to a conic that is tangent to the line [37, Claim, p. 8]. A diagonal in a 3-point regenerates to a conic that is tangent to the other two lines [1]. A 4-point regenerates to two conics and two pairs of parallel lines [37, Figure 3]. A 5-point regenerates as the 4-point regenerates but with an addition of the 5th line [22, Figure 4].

In the next step, a tangent line regenerates into two parallel lines, and each tangency regenerates into three cusps, following the regeneration rules given in [37]. The curve  $S$  is a cuspidal curve with nodes and branch points. A branch point is topologically locally equivalent to  $y^2 + x = 0$  or to

$y^2 - x = 0$ . A node (resp. a cusp) is topologically locally equivalent to  $y^2 - x^2 = 0$  (resp.  $y^2 - x^3 = 0$ ).

We note that 1- and 2-points were considered in [1, 10, 31, 33, 36, 37], 3-points were considered in [1, 2, 3], 4-points were considered in [1, 37], and 5-points were considered in [22]. The regeneration process for large  $k$  can be quite difficult, but work has been done for some specific values: see [22], [12], and [8] for 5-, 6-, and 8-points, respectively.

Now we explain how to derive the related braid monodromy for  $S$  and the fundamental group of its complement in  $\mathbb{CP}^2$ . We will follow the braid monodromy algorithm of Moishezon-Teicher [34, 35]. A detailed treatment can also be found in [1, 12]. Note that the braid group (and the braid monodromy) is very useful to study the projective plane [26].

Consider the setting (Figure 3).  $S$  is an algebraic curve in  $\mathbb{C}^2$ , with  $p = \deg(S)$ . Let  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  be a generic projection onto the first coordinate. Define the fiber  $K(x) = \{y \mid (x, y) \in S\}$  in  $S$  over a fixed point  $x$ , projected to the  $y$ -axis. Define  $N = \{x \mid \#K(x) < p\}$  and  $M' = \{s \in S \mid \pi|_s \text{ is not étale at } s\}$ ; note that  $\pi(M') = N$ . Let  $\{A_j\}_{j=1}^q$  be the set of points of  $M'$  and let  $N = \{x_j\}_{j=1}^q$  be the projection of  $\{A_j\}_{j=1}^q$  on the  $x$ -axis. Recall that  $\pi$  is generic, so we assume that  $\#(\pi^{-1}(x) \cap M') = 1$  for every  $x \in N$ . Let  $E$  (resp.  $D$ ) be a closed disk on the  $x$ -axis (resp. the  $y$ -axis), such that  $M' \subset E \times D$  and  $N \subset \text{Int}(E)$ . We choose  $u \in \partial E$ , a real point far enough from the set  $N$ , so that  $x \ll u$  for every  $x \in N$ . Define  $\mathbb{C}_u = \pi^{-1}(u)$  and number the points of  $K = \mathbb{C}_u \cap S$  as  $\{1, \dots, p\}$ .

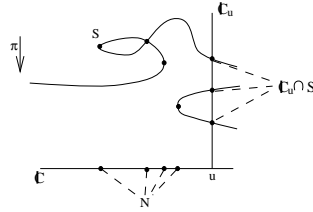


Figure 3: *General setting*

We now construct a g-base for the fundamental group  $\pi_1(E - N, u)$ . Take a set of paths  $\{\gamma_j\}_{j=1}^q$  that connect  $u$  with the points  $\{x_j\}_{j=1}^q$  of  $N$ . Now encircle each  $x_j$  with a small circle,  $c_j$ , oriented counterclockwise. Denote the path segment from  $u$  to the boundary of this circle by  $\gamma'_j$ . We define an element (a loop) in the g-base as  $\delta_j = \gamma'_j c_j \gamma'^{-1}_j$ . Let  $B_p[D, K]$  be the braid group, and let  $H_1, \dots, H_{p-1}$  be its frame (for complete definitions, see [34, Section III.2]). The braid monodromy of  $S$  is a map  $\varphi : \pi_1(E - N, u) \rightarrow B_p[D, K]$  defined as follows (see [14] in detail): every loop in  $E - N$  starting at  $u$  has liftings to a system of  $p$  paths in  $(E - N) \times D$  starting at each point of  $K = 1, \dots, p$ . Projecting them to  $D$ , we obtain  $p$  paths in  $D$  defining a motion  $\{1(t), \dots, p(t)\}$  (for  $0 \leq t \leq 1$ ) of  $p$  points in  $D$  starting and ending at  $K$ . This motion defines a braid in  $B_p[D, K]$ .

By the Artin Theorem [35], for  $j = 1, \dots, q$ , there exists a half-twist  $Z_j \in B_p[D, K]$  and  $\varepsilon_j \in \mathbb{Z}$ , such that  $\varphi(\delta_j) = Z_j^{\varepsilon_j}$ , where  $Z_j$  is a half-twist. The following proposition allows us to determine the associated exponent  $\varepsilon$ ;

**Proposition 2.2.** (Moishezon-Teicher) [34, p. 487 Proposition-Example VI.1.1.] Let  $E = \{x \in \mathbb{C} \mid |x| \leq 1\}$ ,  $D = \{y \in \mathbb{C} \mid y \leq R\}$ ,  $R \gg 1$ , and  $C$  be the curve  $y^2 = x^\varepsilon$ . Denote by  $\varphi : \pi_1(E - N, 1) \rightarrow B_2[D, \{1, -1\}]$  the braid monodromy of  $C$ . Let  $\gamma \in \pi_1(E - N, 1)$  be a loop. Then  $\varphi(\gamma) = h^\varepsilon$ , where  $h$  is the positive half-twist defined by  $[-1, 1]$ .

We now explain how to get the braid monodromy around each singularity in  $S$ , following the notation of Moishezon-Teicher. Let  $A_j$  be a singularity in  $S$  and  $x_j$  its projection by  $\pi$  to the  $x$ -axis. We choose a point  $x'_j$  next to  $x_j$ , such that  $\pi^{-1}(x'_j)$  is a typical fiber. We encircle  $A_j$  with a very small circle in such a way that the typical fiber  $\pi^{-1}(x'_j)$  intersects the circle at two points, say  $a, b$ . We fix a skeleton  $\xi_{x'_j}$  that connects  $a$  and  $b$ , and denote it as  $\langle a, b \rangle$ . The Lefschetz diffeomorphism  $\Psi$  (see [34]) defines the corresponding skeleton  $(\xi_{x'_j})\Psi$  in the typical fiber  $\mathbb{C}_u$ . This one defines a motion of its two endpoints, which induces a half-twist  $Z_j = \Delta\langle(\xi_{x'_j})\Psi\rangle$ . As above,  $\varphi(\delta_j) = \Delta\langle(\xi_{x'_j})\Psi\rangle^{\varepsilon_j}$ . The braid monodromy factorization associated to  $S$  is  $\Delta_p^2 = \prod_{j=1}^q \varphi(\delta_j)$ .

Using the braid monodromy factorization, we compute the fundamental group of the complement of  $S$ . By the van Kampen Theorem [43], there is a "good" geometric base  $\{\Gamma_j\}$  of  $\pi_1(\mathbb{C}_u - S \cap \mathbb{C}_u, *)$ , such that the fundamental group  $\pi_1(\mathbb{CP}^2 - S)$  of the complement of  $S$  in  $\mathbb{CP}^2$  is generated by the images of  $\{\Gamma_j\}$  with the relations  $\varphi(\delta_i)\Gamma_j = \Gamma_j \quad \forall i, j$ .

For our purposes, we take the curve  $S$  to be the branch curve of a smooth surface  $X$ , which is a cuspidal curve with nodes and branch points. Consider a small circle around a singularity. Denote by  $a$  and  $b$  the intersection points of the two branches with this small circle. Note that the branches meet at the singularity. Let  $\Gamma_a, \Gamma_b$  be two non-intersecting loops in  $\pi_1(\mathbb{C}_u - S \cap \mathbb{C}_u, *)$  around the intersection points of the branches with the fiber  $\mathbb{C}_u$  (constructed by cutting each of the paths and creating two loops that proceed along the two parts and encircle  $a$  and  $b$ ); see [34, Proposition-Example VI.1.1]. Then by the van Kampen Theorem, we have the relations  $\langle \Gamma_a, \Gamma_b \rangle = \Gamma_a \Gamma_b \Gamma_a \Gamma_b^{-1} \Gamma_a^{-1} \Gamma_b^{-1} = 1$  for a cusp,  $[\Gamma_a, \Gamma_b] = \Gamma_a \Gamma_b \Gamma_a^{-1} \Gamma_b^{-1} = 1$  for a node, and  $\Gamma_a = \Gamma_b$  for a branch point. These relations, with the addition of the projective relation, generate the group  $\pi_1(\mathbb{CP}^2 - S)$  completely. In this manner, we construct a presentation for the group  $\pi_1(\mathbb{CP}^2 - S)$  by means of generators and relations.

Recall now that  $S$  is of degree  $2m$  (after the regeneration process). Therefore the braids related to  $S$  can be:

- (1)  $Z_{j \ j'}$  for a branch point,
- (2)  $Z_{i,j \ j'}^2$  is a product of two braids  $Z_{i \ j}^2, Z_{i \ j'}^2$  for nodes,
- (3)  $Z_{i,j \ j'}^3$  is a product of three braids  $Z_{i \ j}^3, (Z_{i \ j}^3)^{Z_{j \ j'}}, (Z_{i \ j}^3)^{Z_{j \ j'}^{-1}}$  for cusps.

We denote the generators of the group  $\pi_1(\mathbb{CP}^2 - S)$  as  $\Gamma_1, \Gamma'_1, \dots, \Gamma_{2m}, \Gamma'_{2m}$ . By the van Kampen Theorem [43] we can get a presentation of  $\pi_1(\mathbb{CP}^2 - S)$  by means of generators  $\{\Gamma_j, \Gamma'_j\}$  and relations of the types:

- (1)  $\Gamma_j = \Gamma'_j$  for a branch point in a conic,
- (2)  $[\Gamma_i, \Gamma_j] = [\Gamma_i, \Gamma'_j] = e$  for nodes,
- (3)  $\langle \Gamma_i, \Gamma_j \rangle = \langle \Gamma_i, \Gamma'_j \rangle = \langle \Gamma_i, \Gamma_j^{-1} \Gamma'_j \Gamma_j \rangle = e$  for cusps.

To each list of relations we add the projective relation  $\prod_{j=m}^1 \Gamma'_j \Gamma_j = e$ . Moreover, in some cases in the paper, we have parasitic intersections that contribute commutation relations. These intersections come from lines in  $X_0$  that do not intersect, but when projecting  $X_0$  onto  $\mathbb{CP}^2$ , they will intersect. See details in [33].

Our techniques also allow us to compute fundamental groups of Galois covers. We recall from [33] that if  $f : X \rightarrow \mathbb{CP}^2$  is a generic projection of degree  $n$ , then  $X_{\text{Gal}}$ , the Galois cover, is defined as follows:

$$X_{\text{Gal}} = \overline{(X \times_{\mathbb{CP}^2} \dots \times_{\mathbb{CP}^2} X) - \Delta},$$

where the product is taken  $n$  times, and  $\Delta$  is the diagonal. We apply the theorem of Moishezon-Teicher. There is an exact sequence

$$0 \rightarrow K \rightarrow \pi_1(\mathbb{CP}^2 - S) \rightarrow S_n \rightarrow 0, \quad (1)$$

where the second map takes the generators of  $\pi_1(\mathbb{CP}^2 - S)$  to transpositions in the symmetric group  $S_n$  according to the order of the lines in the degenerated surface. The fundamental group  $\pi_1(X_{\text{Gal}})$  is the quotient of  $K$  by the relations  $\{\Gamma_j^2 = e, \Gamma_j'^2 = e\}$ . Then we obtain a presentation of the fundamental group of the Galois cover, and simplify the relations to produce a canonical presentation that identifies with  $\pi_1(X_{\text{Gal}})$ , using the theory of Coxeter covers of the symmetric groups.

### 3 Calculations of the fundamental group

In this section, we consider the Hirzebruch surface  $F_{1(2,1)}$  (Subsection 3.1), a union of the surface  $\mathbb{CP}^1 \times \mathbb{CP}^1$  and a plane (Subsection 3.2), a union of the Veronese surface  $V_2$  and a plane (Subsection 3.3), two cases of a union of the Cayley surface and two planes (Subsection 3.4), a union of the quartic 4-point surface with a plane (Subsection 3.5), the quintic 5-point surface (Subsection 3.6), and the 4-point quintic degeneration (Subsection 3.7).

**Lemma 3.1.** *There are 8 possible quintic degenerations, corresponding to Figures 4, 5, 6, 8, 10, 12, 13, and 14.*

*Proof.* We construct the degenerations combinatorially by gluing triangles. We add one triangle to quartic degenerations. From the graph theory we get eight possible cases. Figure 4 is from the Hirzebruch surface  $F_{1(2,1)}$ . Figure 5 is the degeneration of  $\mathbb{CP}^1 \times \mathbb{CP}^1$  with a plane. Figure 6 is the degeneration of the Veronese surface with a plane. Figures 8 and 10 are the degenerations of the Cayley surface and two other planes. Figure 12 is the degeneration of the 4-point surface with a

plane. Figure 13 is the degeneration of a 5-point quintic surface. Figure 14 is the degeneration of a 4-point quintic surface.

Every case is different. It is not possible to glue them any further, because in degree 5 this would force us to have three planes meeting in a line.  $\square$

### 3.1 The Hirzebruch surface $F_1(2, 1)$

Let  $F_1$  be the 1-th Hirzebruch surface, i.e., the projection of the vector bundle  $\mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}$ . Denote by  $s$  the holomorphic section of  $\mathcal{O}_{\mathbb{CP}^1}(1)$ , and by  $E_0 \subset F_1$  the image of the section  $(s, 1)$  of  $\mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}$ . The Picard group is always generated by the fiber  $C$  and  $E_0$ . Note that  $2C + E_0$  is very ample and thus defines an embedding  $f_{|2C+E_0|} : F_1 \rightarrow \mathbb{CP}^N$ . Let  $F_1(2, 1) = f_{|2C+E_0|}(F_1)$ . By the constructions in [31],  $F_1(2, 1)$  degenerates to a union of five planes, as depicted in Figure 4.

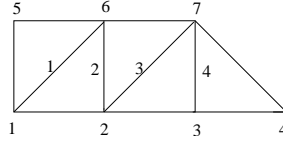


Figure 4: *Degeneration of the Hirzebruch surface  $F_1(2, 1)$*

**Theorem 3.2.** *The fundamental group of the Galois cover  $\pi_1(X_{Gal})$  of the Hirzebruch surface  $F_1(2, 1)$  is trivial.*

*Proof.* See [31, Theorem 0.1]  $\square$

### 3.2 The union of $\mathbb{CP}^1 \times \mathbb{CP}^1$ degeneration and a plane

In this subsection we investigate the surface whose degeneration is depicted in Figure 5, i.e., the union of the  $\mathbb{CP}^1 \times \mathbb{CP}^1$  type degeneration with a plane. In the degeneration, one can see the common edge, numbered as 1.

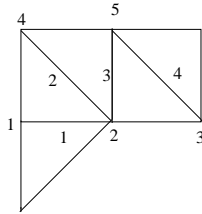


Figure 5: *The union of  $\mathbb{CP}^1 \times \mathbb{CP}^1$  degeneration and a plane*

**Theorem 3.3.**  *$\pi_1(X_{Gal})$  of surfaces with the degeneration as in Figure 5 is trivial.*



*Proof.* The branch curve  $S$  in  $\mathbb{CP}^2$  is an arrangement of 4 lines. We regenerate each vertex in turn and compute group  $G$ .

Vertices 1, 3, and 4 are 1-points; therefore, they give rise to the braids  $Z_{1\ 1'}$ ,  $Z_{4\ 4'}$ , and  $Z_{2\ 2'}$ , respectively, and hence to the following relations in  $G$ :

$$\Gamma_1 = \Gamma'_1, \quad \Gamma_4 = \Gamma'_4, \quad \Gamma_2 = \Gamma'_2. \quad (2)$$

Vertex 5 is a 2-point that gives rise to the braid monodromy factors  $Z_{3\ 3',4}^3, (Z_{4\ 4'})^{Z_{3\ 3',4}^2}$  and to the following relations:

$$\langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma'_3, \Gamma_4 \rangle = \langle \Gamma_3^{-1} \Gamma'_3 \Gamma_3, \Gamma_4 \rangle = e, \quad (3)$$

$$\Gamma'_4 = \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma_4 \Gamma_3^{-1} \Gamma_3'^{-1} \Gamma_4^{-1}. \quad (4)$$

Vertex 2 is a 3-point that regenerates to 2 lines 1, 3 tangent to a conic  $(2, 2')$ . Its braid monodromy factors are:

$$\begin{aligned} \tilde{\Delta}_2 = & (Z_{1\ 3}^2)^{Z_{11',2}^2} \cdot (Z_{1'\ 3}^2)^{Z_{11',2}^2} \cdot (Z_{1\ 3'}^2)^{Z_{11',2}^2} \cdot (Z_{1'\ 3'}^2)^{Z_{11',2}^2} \\ & \cdot (Z_{2\ 2'})^{Z_{1\ 1',2}^2 Z_{2',3\ 3'}^2} \cdot Z_{2',3\ 3'}^3 \cdot Z_{1\ 1',2}^3. \end{aligned} \quad (5)$$

$\tilde{\Delta}_2$  gives rise the following relations in  $G$ :

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma'_1, \Gamma_2 \rangle = \langle \Gamma_1^{-1} \Gamma'_1 \Gamma_1, \Gamma_2 \rangle = e, \quad (6)$$

$$\langle \Gamma'_2, \Gamma_3 \rangle = \langle \Gamma'_2, \Gamma'_3 \rangle = \langle \Gamma'_2, \Gamma_3^{-1} \Gamma'_3 \Gamma_3 \rangle = e, \quad (7)$$

$$\Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_3^{-1} \Gamma_3'^{-1} = \Gamma_2 \Gamma'_1 \Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma_1'^{-1} \Gamma_2^{-1}, \quad (8)$$

$$[\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma_3] = [\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma'_3] = [\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma_3] = [\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma'_3] = e. \quad (9)$$

We also have the following parasitic and projective relations:

$$[\Gamma_1, \Gamma_4] = [\Gamma_1, \Gamma'_4] = [\Gamma'_1, \Gamma_4] = [\Gamma'_1, \Gamma'_4] = e, \quad (10)$$

$$[\Gamma_2, \Gamma_4] = [\Gamma_2, \Gamma'_4] = [\Gamma'_2, \Gamma_4] = [\Gamma'_2, \Gamma'_4] = e, \quad (11)$$

$$\Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_1 = e. \quad (12)$$

After simplification, we have the following relations in  $G$ :

$$\Gamma_1 = \Gamma'_1, \quad \Gamma_2 = \Gamma'_2, \quad \Gamma_4 = \Gamma'_4, \quad \Gamma'_3 = \Gamma_4^{-2} \Gamma_3 \Gamma_4^2,$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = e,$$

$$[\Gamma_1, \Gamma_4] = [\Gamma_2, \Gamma_4] = [\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma_3] = e,$$

$$\Gamma_3 \Gamma_4^2 \Gamma_3 \Gamma_2^2 \Gamma_1 = e.$$

It is easy to see that  $[\Gamma_2\Gamma_1\Gamma_2^{-1}, \Gamma_4] = e$  and  $\langle \Gamma_2\Gamma_1\Gamma_2^{-1}, \Gamma_2 \rangle = e$ . Let  $\{\Gamma_2\Gamma_1\Gamma_2^{-1}, \Gamma_2, \Gamma_3, \Gamma_4\}$  be the generators of  $G/\langle \Gamma_1^2, \Gamma_2^2, \Gamma_3^2, \Gamma_4^2 \rangle$ , then  $G/\langle \Gamma_1^2, \Gamma_2^2, \Gamma_3^2, \Gamma_4^2 \rangle = \{\Gamma_2\Gamma_1\Gamma_2^{-1}, \Gamma_2, \Gamma_3, \Gamma_4 \mid S_5 \text{ type relations}\} \cong S_5$ , hence  $\pi_1(X_{\text{Gal}})$  is trivial.

This completes the proof.  $\square$

### 3.3 The union of the Veronese $V_2$ degeneration and a plane

We denote by  $V_2$  the Veronese surface of order 2. In this subsection we introduce the degeneration of a union of the Veronese surface  $V_2$  and a plane. The degeneration of this surface is a union of five planes, where  $V_2$  and the plane are united along the edge, numbered as 4, see Figure 6.

We note that the surface  $V_2$  is atypical in different algebraic-geometrical theories, for example, it is the exception case for Chisini's conjecture [21].

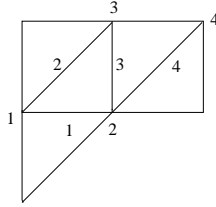


Figure 6: *The union of  $V_2$  degeneration and a plane*

**Theorem 3.4.**  $\pi_1(X_{\text{Gal}})$  of surfaces with the degeneration as in Figure 6 is  $\mathbb{Z}^8$ .

*Proof.* The branch curve  $S$  in  $\mathbb{CP}^2$  is an arrangement of 4 lines. We regenerate each vertex in turn and compute group  $G$ .

Vertex 4 is a 1-point that gives rise to braid  $Z_{4\ 4'}$ , and derives the following relation in  $G$ :

$$\Gamma_4 = \Gamma'_4. \quad (13)$$

Vertex 1 (resp. 3) is a 2-point that gives rise to the braid monodromy factors  $Z_{1\ 1',2}^3$  and  $(Z_{2\ 2'})^{Z_{1\ 1',2}^2}$  (resp.  $Z_{2',3\ 3'}^3$  and  $(Z_{2\ 2'})^{Z_{2',3\ 3'}^2}$ ). These braids yield the following relations:

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma'_1, \Gamma_2 \rangle = \langle \Gamma_1^{-1} \Gamma'_1 \Gamma_1, \Gamma_2 \rangle = e, \quad (14)$$

$$\Gamma'_2 = \Gamma_2 \Gamma'_1 \Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma'_1^{-1} \Gamma_2^{-1}, \quad (15)$$

$$\langle \Gamma'_2, \Gamma_3 \rangle = \langle \Gamma'_2, \Gamma'_3 \rangle = \langle \Gamma'_2, \Gamma_3^{-1} \Gamma'_3 \Gamma_3 \rangle = e, \quad (16)$$

$$\Gamma_2 = \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_3^{-1} \Gamma'_3^{-1}. \quad (17)$$

Vertex 2 is a 3-point (the intersection of the lines 1, 3, 4) that regenerates to a line 3 tangent to 2 conics  $(1, 1'), (4, 4')$ . The braid monodromy corresponding to this 3-point is:

$$\tilde{\Delta} = Z_{14}^2 \cdot Z_{1'4}^2 \cdot Z_{14'}^2 \cdot Z_{1'4'}^2 \cdot Z_{1',3,3'}^3 \cdot (Z_{11'})^{Z_{1',3,3'}} \cdot (Z_{33',4}^3)^{Z_{1',3,3'}} \cdot (Z_{44'})^{Z_{33',4}^2 Z_{1',3,3'}}.$$

$\tilde{\Delta}$  thus gives rise to the following relations:

$$\langle \Gamma'_1, \Gamma_3 \rangle = \langle \Gamma'_1, \Gamma'_3 \rangle = \langle \Gamma'_1, \Gamma_3^{-1} \Gamma'_3 \Gamma_3 \rangle = e, \quad (18)$$

$$\Gamma_1 = \Gamma'_3 \Gamma_3 \Gamma'_1 \Gamma_3^{-1} \Gamma'_3^{-1}, \quad (19)$$

$$\langle \Gamma_4, \Gamma'_3 \Gamma_3 \Gamma'_1 \Gamma_3 \Gamma_1^{-1} \Gamma_3^{-1} \Gamma'_3^{-1} \rangle = \langle \Gamma_4, \Gamma'_3 \Gamma_3 \Gamma'_1 \Gamma'_3 \Gamma_1^{-1} \Gamma_3^{-1} \Gamma'_3^{-1} \rangle = \langle \Gamma_4, \Gamma'_3 \Gamma_3 \Gamma'_1 \Gamma_3^{-1} \Gamma'_3 \Gamma_3 \Gamma_1^{-1} \Gamma_3^{-1} \Gamma'_3^{-1} \rangle = e, \quad (20)$$

$$\Gamma'_4 = \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_1 \Gamma'_3 \Gamma_3 \Gamma_1^{-1} \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_1 \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_1^{-1} \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_4^{-1}, \quad (21)$$

$$[\Gamma_1, \Gamma_4] = [\Gamma_1, \Gamma'_4] = [\Gamma'_1, \Gamma_4] = [\Gamma'_1, \Gamma'_4] = e. \quad (22)$$

We also have the following parasitic and projective relations:

$$[\Gamma_2, \Gamma_4] = [\Gamma_2, \Gamma'_4] = [\Gamma'_2, \Gamma_4] = [\Gamma'_2, \Gamma'_4] = e, \quad (23)$$

$$\Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_1 = e. \quad (24)$$

By (13), (19), and (21), we have

$$\Gamma_4 = \Gamma_1 \Gamma'_3 \Gamma_3 \Gamma_1^{-1} \Gamma_4 \Gamma_1 \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_1^{-1}.$$

As  $[\Gamma_1, \Gamma_4] = e$ , we obtain  $[\Gamma'_3 \Gamma_3, \Gamma_4] = e$ .

By (13), (15), and (19), we can choose  $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma'_3, \Gamma_4$  to be the generators of  $G$ .

After simplification, we have the following relations in  $G$ :

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_3^{-1} \Gamma'_3 \Gamma_1 \Gamma'_3 \Gamma_3 \rangle = \langle \Gamma_2, \Gamma'_3 \Gamma_3 \Gamma_1 \Gamma_3^{-1} \Gamma'_3^{-1} \rangle = e, \quad (25)$$

$$\langle \Gamma_2, \Gamma'_3 \rangle = \langle \Gamma_2, \Gamma'_3 \Gamma_3 \Gamma'_3^{-1} \rangle = \langle \Gamma_2, \Gamma'_3 \Gamma_3 \Gamma'_3 \Gamma_3^{-1} \Gamma'_3^{-1} \rangle = e, \quad (26)$$

$$\langle \Gamma_1, \Gamma'_3 \rangle = \langle \Gamma_1, \Gamma'_3 \Gamma_3 \Gamma'_3^{-1} \rangle = \langle \Gamma_1, \Gamma'_3 \Gamma_3 \Gamma'_3 \Gamma_3^{-1} \Gamma'_3^{-1} \rangle = e, \quad (27)$$

$$\langle \Gamma_4, \Gamma'_3 \rangle = \langle \Gamma_4, \Gamma'_3 \Gamma_3 \Gamma'_3^{-1} \rangle = \langle \Gamma_4, \Gamma'_3 \Gamma_3 \Gamma'_3 \Gamma_3^{-1} \Gamma'_3^{-1} \rangle = e, \quad (28)$$

$$[\Gamma_1, \Gamma_4] = [\Gamma_2, \Gamma_4] = [\Gamma'_3 \Gamma_3, \Gamma_4] = e, \quad (29)$$

$$\Gamma_2 \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_1 \Gamma'_3 \Gamma_3 \Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_1^{-1} \Gamma'_3 \Gamma_3 \Gamma_2^{-1} = \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_2 \Gamma'_3 \Gamma_3, \quad (30)$$

$$\Gamma_4^2 \Gamma_2 \Gamma'_3 \Gamma_3 \Gamma_2 \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_1 \Gamma'_3 \Gamma_3 \Gamma_1 = e. \quad (31)$$

We define the surjective map  $\varphi : G / \langle \Gamma_i^2, \Gamma_i'^2 \rangle \rightarrow S_5$  by

$$\Gamma_1 \mapsto (13), \quad \Gamma_2 \mapsto (23), \quad \Gamma_3 \mapsto (34), \quad \Gamma'_3 \mapsto (34), \quad \Gamma_4 \mapsto (45).$$

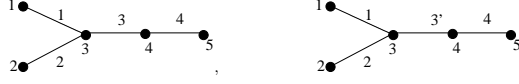


Figure 7: *Diagram for relations in  $G$*

By [39, pp. 141], the generators of the kernel are  $(\Gamma_3\Gamma_1\Gamma_2\Gamma_1)^2$ ,  $(\Gamma'_3\Gamma_1\Gamma_2\Gamma_1)^2$  and their conjugates. Because  $S_5$  is generated by 4 transpositions, we need to take 4 conjugates of  $(\Gamma_3\Gamma_1\Gamma_2\Gamma_1)^2$ ,  $(\Gamma'_3\Gamma_1\Gamma_2\Gamma_1)^2$ .

Hence,  $\pi_1(X_{\text{Gal}}) \cong \ker \varphi \cong \mathbb{Z}^8$ .

This completes the proof.  $\square$

### 3.4 The union of the Cayley degeneration and two planes

The classification of singular cubic surfaces in  $\mathbb{CP}^3$  was done in the 1860s, by Schläfli [41] and Cayley [20]. Surface XVI in Cayley's classification is now called the Cayley cubic, and when embedded in  $\mathbb{CP}^3$ , it is defined by the following equation:

$$4(X^3 + Y^3 + Z^3 + W^3) - (X + Y + Z + W)^3 = 0.$$

It has four singularities, which are ordinary double points. Cayley noticed that this surface is a unique cubic surface having four ordinary double points, which is the maximal possible number of double points for a cubic surface (see, for example, Salmon's book [40]).

In this subsection we introduce two cases of a degeneration that is a union of the Cayley surface and two planes. We call them Type I (Figure 8) and Type II (Figure 10).

#### 3.4.1 Type I

In this subsection we consider the union of the Cayley surface and two planes (Type I). The degeneration of this surface is a union of five planes, where the common edges of the Cayley degeneration and the two planes are 4 and 5, see Figure 8.

The Hilbert scheme of del Pezzo surfaces of degree 5 contains configurations of planes as in Figure 8 (degenerations of Type I). Indeed, the rational normal scroll  $F$  degenerates to the 4 “top” planes in Figure 8 (this is a toric degeneration, corresponding to the subdivision of the  $(1, 2)$ -rectangle as Figure 16). Then the “bottom” length 2 polyline 34 in Figure 8 is a conic  $C$ , which degenerates here in two lines corresponding to the segments (edges) 2 and 3. The plane  $P$  of  $C$  becomes the plane spanned by edges 2 and 3, which is just the “bottom plane” in Figure 8.

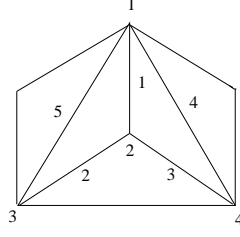


Figure 8: *Degeneration of Type I*

The Cayley surface, its degeneration, and the related braid monodromy of the 3-points appearing there, were computed in [3, 9]. Here we have the same computations for the 3-points with modification of numbers, therefore we give the relations in group  $G$ , without the list of braids.

**Theorem 3.5.**  $\pi_1(X_{Gal})$  of surfaces with the degeneration as in Figure 8 is  $(\mathbb{Z}_2^2 \times \mathbb{Z}^8) \oplus (\mathbb{Z}_2^2 \times \mathbb{Z}^8)$ .

*Proof.* The branch curve  $S$  in  $\mathbb{CP}^2$  is an arrangement of 5 lines. We regenerate each vertex in turn and compute group  $G$ .

Vertex 4 (resp. 3) is a 2-point. The braid monodromy factors are  $Z_{3',4'}^3$  and  $Z_{33'}^{Z_{3',4'}^2}$  (resp.  $Z_{2',5'}^3$  and  $Z_{22'}^{Z_{2',5'}^2}$ ), giving rise to the following relations in  $G$ :

$$\langle \Gamma'_3, \Gamma_4 \rangle = \langle \Gamma'_3, \Gamma'_4 \rangle = \langle \Gamma'_3, \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \rangle = e, \quad (32)$$

$$\Gamma_3 = \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_4^{-1} \Gamma'_4^{-1}, \quad (33)$$

$$\langle \Gamma'_2, \Gamma_5 \rangle = \langle \Gamma'_2, \Gamma'_5 \rangle = \langle \Gamma'_2, \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \rangle = e, \quad (34)$$

$$\Gamma_2 = \Gamma'_5 \Gamma_5 \Gamma'_2 \Gamma_5^{-1} \Gamma'_5^{-1}. \quad (35)$$

Vertex 1 is a 3-point that regenerates to a line 1 tangent to two conics  $(4, 4'), (5, 5')$ . The braid monodromy factors are the following:

$$\tilde{\Delta}_1 = Z_{4'5}^2 \cdot Z_{4'5}^2 \cdot \bar{Z}_{4'5'}^2 \cdot (Z_{4'5'}^2)^{Z_{55'}^{-2}} \cdot Z_{1'1',5}^3 \cdot (Z_{5'5'})^{Z_{1'1',5}^2} \cdot Z_{1'1',4}^3 \cdot (Z_{4'4'})^{Z_{1'1',4}^2}.$$

$\tilde{\Delta}_1$  gives rise to the following relations in  $G$ :

$$[\Gamma_4, \Gamma_5] = [\Gamma'_4, \Gamma_5] = [\Gamma_4, \Gamma_5^{-1} \Gamma'_5 \Gamma_5] = [\Gamma'_4, \Gamma_5^{-1} \Gamma'_5 \Gamma_5] = e, \quad (36)$$

$$\langle \Gamma_1, \Gamma_5 \rangle = \langle \Gamma'_1, \Gamma_5 \rangle = \langle \Gamma_1^{-1} \Gamma'_1 \Gamma_1, \Gamma_5 \rangle = e, \quad (37)$$

$$\Gamma'_5 = \Gamma_5 \Gamma'_1 \Gamma_1 \Gamma_5 \Gamma_1^{-1} \Gamma'_1^{-1} \Gamma_5^{-1}, \quad (38)$$

$$\langle \Gamma_1, \Gamma_4 \rangle = \langle \Gamma'_1, \Gamma_4 \rangle = \langle \Gamma_1^{-1} \Gamma'_1 \Gamma_1, \Gamma_4 \rangle = e, \quad (39)$$

$$\Gamma'_4 = \Gamma_4 \Gamma'_1 \Gamma_1 \Gamma_4 \Gamma_1^{-1} \Gamma'_1^{-1} \Gamma_4^{-1}. \quad (40)$$

Vertex 2 is also a 3-point of Cayley type, the braid monodromy factors were computed in [3], and give rise to the following relations:

$$\Gamma_2 = \Gamma'_2, \quad (41)$$

$$[\Gamma'_1, \Gamma_2^{-1} \Gamma_3 \Gamma'_2] = [\Gamma'_1, \Gamma_2^{-1} \Gamma_3^{-1} \Gamma'_3 \Gamma_3 \Gamma'_2] = [\Gamma_1, \Gamma_2^{-1} \Gamma_3 \Gamma'_2] = [\Gamma_1, \Gamma_2^{-1} \Gamma_3^{-1} \Gamma'_3 \Gamma_3 \Gamma'_2] = e, \quad (42)$$

$$\langle \Gamma_1, \Gamma'_2 \rangle = \langle \Gamma'_1, \Gamma'_2 \rangle = \langle \Gamma_1^{-1} \Gamma_1 \Gamma'_1, \Gamma'_2 \rangle = e, \quad (43)$$

$$\Gamma'_2 = \Gamma_1^{-1} \Gamma_1^{-1} \Gamma_2^{-1} \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2^{-1} \Gamma_3^{-1} \Gamma_3^{-1} \Gamma'_2 \Gamma_1 \Gamma_1, \quad (44)$$

$$\langle \Gamma'_2 \Gamma_2 \Gamma_2^{-1}, \Gamma_3 \rangle = \langle \Gamma'_2 \Gamma_2 \Gamma_2^{-1}, \Gamma'_3 \rangle = \langle \Gamma'_2 \Gamma_2 \Gamma_2^{-1}, \Gamma'_3 \Gamma_3 \Gamma_3^{-1} \rangle = e. \quad (45)$$

We also have the following parasitic and projective relations:

$$[\Gamma_2, \Gamma_4] = [\Gamma_2, \Gamma'_4] = [\Gamma'_2, \Gamma_4] = [\Gamma'_2, \Gamma'_4] = e, \quad (46)$$

$$[\Gamma_3, \Gamma_5] = [\Gamma_3, \Gamma'_5] = [\Gamma'_3, \Gamma_5] = [\Gamma'_3, \Gamma'_5] = e, \quad (47)$$

$$\Gamma'_5 \Gamma_5 \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_1 = e. \quad (48)$$

By (32) and (33) we get  $\langle \Gamma_3, \Gamma_4 \rangle = e$ , and by (34) and (35) we get  $\langle \Gamma_2, \Gamma_5 \rangle = e$ .

By (45) and (41) we get

$$\langle \Gamma_2, \Gamma_3 \rangle = e. \quad (49)$$

It is easy to see that in  $G$ , we can choose  $\{\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$  to be the generators. Then  $G/\langle \Gamma_i^2 \rangle$  ( $i = 1, 1', \dots, 5, 5'$ ) is generated by  $\{\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$  with the following relations:

$$\Gamma_1^2 = \Gamma_1'^2 = \Gamma_2^2 = \Gamma_3^2 = \Gamma_4^2 = \Gamma_5^2 = e, \quad (50)$$

$$[\Gamma_2, \Gamma_4] = [\Gamma_3, \Gamma_5] = [\Gamma_4, \Gamma_5] = e, \quad (51)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_1, \Gamma_4 \rangle = \langle \Gamma_1, \Gamma_5 \rangle = e, \quad (52)$$

$$\langle \Gamma'_1, \Gamma_2 \rangle = \langle \Gamma'_1, \Gamma_4 \rangle = \langle \Gamma'_1, \Gamma_5 \rangle = e,$$

$$\langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_5 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = e, \quad (53)$$

$$[\Gamma_1, \Gamma_2^{-1} \Gamma_3 \Gamma_2] = [\Gamma'_1, \Gamma_2^{-1} \Gamma_3 \Gamma_2] = e. \quad (54)$$

We define the surjective map  $\rho : G/\langle \Gamma_i^2, \Gamma_i'^2 \rangle \rightarrow S_5$  according to Figure 8 by

$$\Gamma_1 \mapsto (1\ 2), \Gamma'_1 \mapsto (1\ 2), \Gamma_2 \mapsto (1\ 3), \Gamma_3 \mapsto (2\ 3), \Gamma_4 \mapsto (2\ 4), \Gamma_5 \mapsto (1\ 5).$$

Then the diagrams for the above map  $\rho$  are as shown in Figure 9.



Figure 9: *Diagrams for  $\rho$*

By [39, pp.141], the subgroup of  $\ker \rho$  contributed by the left part of Figure 9 is generated by  $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_2, \Gamma_2 \Gamma_3 \Gamma_1 \Gamma_3, \Gamma_3 \Gamma_1 \Gamma_2 \Gamma_1$ , arising from the cycle  $\Gamma_1, \Gamma_2, \Gamma_3$ , and by the conjugates of

$(\Gamma_2\Gamma_1\Gamma_5\Gamma_1)^2$  and  $(\Gamma_3\Gamma_1\Gamma_4\Gamma_1)^2$ , arising from the relations involving three edges meeting at a vertex.

Because the cycle only involves three vertices and because  $S_3$  is generated by two transpositions, it is enough to take the preimages of two transpositions that arise from the cyclic relation. In particular, all elements of the kernel that arise from the cyclic relation are generated by  $\Gamma_1\Gamma_2\Gamma_3\Gamma_2$ ,  $\Gamma_2\Gamma_3\Gamma_1\Gamma_3$  and their conjugates. It is easy to see that they have order 2 and commute. Then the subgroup generated by these 2 elements is  $\mathbb{Z}_2^2$ .

By [39, pp. 141], the generators of the kernel that arise from three edges meeting at a vertex are  $(\Gamma_2\Gamma_1\Gamma_5\Gamma_1)^2$ ,  $(\Gamma_3\Gamma_1\Gamma_4\Gamma_1)^2$  and their conjugates. Because  $S_5$  is generated by 4 transpositions, we need to take 4 conjugates of  $(\Gamma_2\Gamma_1\Gamma_5\Gamma_1)^2$ ,  $(\Gamma_3\Gamma_1\Gamma_4\Gamma_1)^2$ .

Hence,  $\pi_1(X_{\text{Gal}}) = \ker \rho \cong (\mathbb{Z}_2^2 \times \mathbb{Z}^8) \oplus (\mathbb{Z}_2^2 \times \mathbb{Z}^8)$ .

This completes the proof. □

### 3.4.2 Type II

In this subsection we consider the union of the Cayley surface and two planes (Type II). The degeneration of this surface is a union of five planes, where the common edges of the Cayley degeneration and one of the planes is 3, and two planes have a common edge that is 5, see Figure 10.

The Hilbert scheme of del Pezzo surfaces of degree 5 contains configurations of planes as in Figure 10 (degenerations of Type II). Indeed, the Veronese  $V$  degenerates to the 4 "rightmost" planes (except a plane with vertices 123) in Figure 10 (this is a toric degeneration). Then in vertex 3, the length 2 line 13 of  $V$  corresponds to a conic  $C$ , which degenerates here in two lines corresponding to the segments (edges) 1 and 2. The plane  $P$  of  $C$  becomes the plane spanned by edges 1 and 2, which is just a plane with vertices 123 in Figure 10.

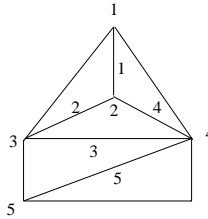


Figure 10: *Degeneration of Type II*

**Theorem 3.6.**  $\pi_1(X_{\text{Gal}})$  of surfaces with the degeneration as in Figure 10 is  $\mathbb{Z}_2^2 \times \mathbb{Z}^4$ .

*Proof.* The branch curve  $S$  in  $\mathbb{CP}^2$  is an arrangement of 5 lines. We regenerate each vertex in turn and compute group  $G$ .

Vertices 1 and 5 give rise to the braids  $Z_{11'}$  and  $Z_{55'}$  respectively, and hence to the relations:

$$\Gamma_1 = \Gamma'_1, \quad \Gamma_5 = \Gamma'_5. \quad (55)$$

Vertex 3 is a 2-point with the braid monodromy factors being  $Z_{2',33'}^3$  and  $(Z_{22'})^{Z_{2',33'}^2}$ . Then we get the following relations:

$$\langle \Gamma'_2, \Gamma_3 \rangle = \langle \Gamma'_2, \Gamma'_3 \rangle = \langle \Gamma'_2, \Gamma_3^{-1} \Gamma'_3 \Gamma_3 \rangle = e, \quad (56)$$

$$\Gamma_2 = \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_3^{-1} \Gamma'_3^{-1}. \quad (57)$$

Vertex 2 is also a 3-point of Cayley type, the braid monodromy factors were computed in [3], and give rise to the following relations:

$$\Gamma_2 = \Gamma'_2, \quad (58)$$

$$[\Gamma'_1, \Gamma_2^{-1} \Gamma_4 \Gamma'_2] = [\Gamma'_1, \Gamma_2^{-1} \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \Gamma'_2] = [\Gamma_1, \Gamma_2^{-1} \Gamma_4 \Gamma'_2] = [\Gamma_1, \Gamma_2^{-1} \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \Gamma'_2] = e, \quad (59)$$

$$\langle \Gamma_1, \Gamma'_2 \rangle = \langle \Gamma'_1, \Gamma'_2 \rangle = \langle \Gamma_1^{-1} \Gamma_1 \Gamma'_1, \Gamma'_2 \rangle = e, \quad (60)$$

$$\Gamma'_2 = \Gamma_1^{-1} \Gamma_1^{-1} \Gamma'_2 \Gamma'_4 \Gamma_4 \Gamma'_2 \Gamma_2^{-1} \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \Gamma'_1 \Gamma_1, \quad (61)$$

$$\langle \Gamma'_2 \Gamma_2 \Gamma_2^{-1}, \Gamma_4 \rangle = \langle \Gamma'_2 \Gamma_2 \Gamma_2^{-1}, \Gamma'_4 \rangle = \langle \Gamma'_2 \Gamma_2 \Gamma_2^{-1}, \Gamma'_4 \Gamma_4 \Gamma'_4^{-1} \rangle = e. \quad (62)$$

Vertex 4 is a 3-point regenerating to a line 3 tangent to two conics  $(4, 4')$ ,  $(5, 5')$  (as line 4, 5 are diagonals) with the braid monodromy factors giving rise to the following relations in  $G$ :

$$[\Gamma_4, \Gamma_5] = [\Gamma'_4, \Gamma_5] = [\Gamma_4, \Gamma_5^{-1} \Gamma'_5 \Gamma_5] = [\Gamma'_4, \Gamma_5^{-1} \Gamma'_5 \Gamma_5] = e, \quad (63)$$

$$\langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma'_3, \Gamma_5 \rangle = \langle \Gamma_3^{-1} \Gamma'_3 \Gamma_3, \Gamma_5 \rangle = e, \quad (64)$$

$$\Gamma'_5 = \Gamma_5 \Gamma'_3 \Gamma_3 \Gamma_5 \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_5^{-1}, \quad (65)$$

$$\langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma'_3, \Gamma_4 \rangle = \langle \Gamma_3^{-1} \Gamma'_3 \Gamma_3, \Gamma_4 \rangle = e, \quad (66)$$

$$\Gamma'_4 = \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma_4 \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_4^{-1}. \quad (67)$$

We also have the following parasitic and projective relations:

$$[\Gamma_1, \Gamma_3] = [\Gamma'_1, \Gamma_3] = [\Gamma_1, \Gamma'_3] = [\Gamma'_1, \Gamma'_3] = e, \quad (68)$$

$$[\Gamma_1, \Gamma_5] = [\Gamma'_1, \Gamma_5] = [\Gamma_1, \Gamma'_5] = [\Gamma'_1, \Gamma'_5] = e, \quad (69)$$

$$[\Gamma_2, \Gamma_5] = [\Gamma'_2, \Gamma_5] = [\Gamma_2, \Gamma'_5] = [\Gamma'_2, \Gamma'_5] = e, \quad (70)$$

$$\Gamma'_5 \Gamma_5 \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_1 = e. \quad (71)$$

By the relations above, we can choose  $\Gamma_i$ ,  $i = 1, \dots, 5$  to be generators of  $G$ . The relations in  $G/\langle \Gamma_i^2 \rangle$  are the following:

$$[\Gamma_1, \Gamma_3] = [\Gamma_1, \Gamma_5] = [\Gamma_2, \Gamma_5] = [\Gamma_1, \Gamma_2 \Gamma_4 \Gamma_2] = [\Gamma_4, \Gamma_5] = e, \quad (72)$$



$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_5 \rangle = e. \quad (73)$$

We define the surjective map  $\rho' : G/\langle \Gamma_i^2 \rangle \rightarrow S_5$  by

$$\Gamma_1 \mapsto (12), \Gamma_2 \mapsto (13), \Gamma_3 \mapsto (34), \Gamma_4 \mapsto (23), \Gamma_5 \mapsto (45).$$

The diagram for the map  $\rho'$  is Figure 11.

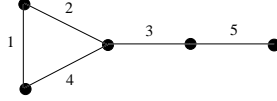


Figure 11: *Diagram for  $\rho'$*

By [39, pp. 141],  $\ker \rho'$  is generated by  $\Gamma_1 \Gamma_2 \Gamma_4 \Gamma_2$ ,  $\Gamma_2 \Gamma_4 \Gamma_1 \Gamma_4$ ,  $\Gamma_4 \Gamma_1 \Gamma_2 \Gamma_1$ , arising from the cycle  $\Gamma_1, \Gamma_2, \Gamma_4$ , and by the conjugates of  $(\Gamma_2 \Gamma_3 \Gamma_4 \Gamma_3)^2$ , arising from the relation involving three edges meeting at a vertex.

Because the cycle only involves three vertices and because  $S_3$  is generated by two transpositions, it is enough to take the preimages of two transpositions that arise from the cyclic relation. In particular, all elements of the kernel that arise from the cyclic relation are generated by  $\Gamma_1 \Gamma_2 \Gamma_4 \Gamma_2$ ,  $\Gamma_2 \Gamma_4 \Gamma_1 \Gamma_4$  and their conjugates. It is easy to see that they have order 2 and commute. Then the subgroup generated by these 2 elements is  $\mathbb{Z}_2^2$ .

By [39, pp. 141], the generators of the kernel that arise from three edges meeting at a vertex are  $(\Gamma_2 \Gamma_3 \Gamma_4 \Gamma_3)^2$  and its conjugates. Because  $S_5$  is generated by 4 transpositions, we need to take 4 conjugates of  $(\Gamma_2 \Gamma_3 \Gamma_4 \Gamma_3)^2$ .

$$\text{Hence, } \pi_1(X_{\text{Gal}}) = \ker \rho' \cong \mathbb{Z}_2^2 \ltimes \mathbb{Z}^4.$$

This completes the proof.  $\square$

### 3.5 A union of the 4-point quartic degeneration and a plane

In this subsection, we take a degeneration of a quartic surface to a plane arrangement with a 4-point.

The Hilbert scheme of del Pezzo surfaces of degree 5 contains reducible surfaces that consist in a general degree 4 complete intersection  $F$  of type  $(2, 2)$  in  $\mathbb{P}^4$  (which is itself a del Pezzo surface), plus a plane  $P$  meeting  $F$  along a line.

We can show that the configuration shown in Figure 12 is a limit of smooth del Pezzo surfaces of degree 5. Indeed, the surface  $F$  can degenerate to the union of 4 planes filling up the subdivided square in Figure 12: Simply degenerate the two quadrics cutting out  $F$  in two general quadric cones with the same vertex (which is then the 4-tuple point  $p$  that is common to the 4 planes). In this degeneration the lines contained in  $F$ , which are 16, are mapped to the following configuration of lines: Take a general quadric  $Q$  in  $\mathbb{P}^4$ , it cuts each of the 4 lines through  $p$  in two points; then on each of the 4 planes take the four lines pairwise joining the pairs of points not on the same line through

$p$ , these are the 16 limits in question. Then a general plane through a line on  $F$  goes to a plane like the “rightmost top” plane in Figure 12.

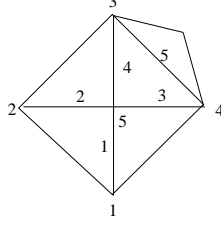


Figure 12: A union of the 4-point quartic degeneration and a plane

**Theorem 3.7.**  $\pi_1(X_{Gal})$  of surfaces with the degeneration as in Figure 12 is trivial.

*Proof.* The branch curve  $S$  in  $\mathbb{CP}^2$  is an arrangement of 5 lines. We regenerate each vertex in turn and compute group  $G$ .

Vertices 1 and 2 are 1-points, giving the braids  $Z_{11'}$  and  $Z_{22'}$  respectively, and hence the following relations in  $G$ :

$$\Gamma_1 = \Gamma'_1, \quad \Gamma_2 = \Gamma'_2. \quad (74)$$

Vertex 3 is a 2-point and it gives the braid monodromy factors  $Z_{44',5}^3$  and  $(Z_{55'})^{Z_{44',5}^2}$ . The relations in  $G$  are:

$$\langle \Gamma_4, \Gamma_5 \rangle = \langle \Gamma'_4, \Gamma_5 \rangle = \langle \Gamma_4^{-1} \Gamma'_4 \Gamma_4, \Gamma_5 \rangle = e, \quad (75)$$

$$\Gamma'_5 = \Gamma_5 \Gamma'_4 \Gamma_4 \Gamma_5 \Gamma_4^{-1} \Gamma'_4^{-1} \Gamma_5^{-1}. \quad (76)$$

Vertex 4 is also a 2-point and it gives the braid monodromy factors  $Z_{33',5}^3$  and  $(Z_{55'})^{Z_{33',5}^2}$ . The relations in  $G$  are:

$$\langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma'_3, \Gamma_5 \rangle = \langle \Gamma_3^{-1} \Gamma'_3 \Gamma_3, \Gamma_5 \rangle = e, \quad (77)$$

$$\Gamma'_5 = \Gamma_5 \Gamma'_3 \Gamma_3 \Gamma_5 \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_5^{-1}. \quad (78)$$

The braid monodromy factors corresponding to the 4-point (vertex 5) were computed in [10]. These braids give rise to the following relations in  $G$ :

$$\langle \Gamma'_1, \Gamma_2 \rangle = \langle \Gamma'_1, \Gamma'_2 \rangle = \langle \Gamma'_1, \Gamma_2^{-1} \Gamma'_2 \Gamma_2 \rangle = e, \quad (79)$$

$$\langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma'_3, \Gamma_4 \rangle = \langle \Gamma_3^{-1} \Gamma'_3 \Gamma_3, \Gamma_4 \rangle = e, \quad (80)$$

$$[\Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_2^{-1} \Gamma'_2^{-1}, \Gamma_4] = e, \quad (81)$$

$$[\Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_2^{-1} \Gamma'_2^{-1}, \Gamma_3^{-1} \Gamma'_3^{-1} \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_3] = e, \quad (82)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_1, \Gamma'_2 \rangle = \langle \Gamma_1, \Gamma_2^{-1} \Gamma'_2 \Gamma_2 \rangle = e, \quad (83)$$

$$\langle \Gamma_3, \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \rangle = \langle \Gamma'_3, \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \rangle = \langle \Gamma_3^{-1} \Gamma'_3 \Gamma_3, \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \rangle = e, \quad (84)$$

$$[\Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma'_2^{-1}, \Gamma_4^{-1} \Gamma'_4 \Gamma_4] = e, \quad (85)$$

$$[\Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_2'^{-1}, \Gamma_3^{-1} \Gamma_3'^{-1} \Gamma_4^{-1} \Gamma_4'^{-1} \Gamma_4 \Gamma_4' \Gamma_3 \Gamma_3'] = e, \quad (86)$$

$$\Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2 \Gamma_1'^{-1} \Gamma_2^{-1} \Gamma_2'^{-1} = \Gamma_4 \Gamma_3 \Gamma_4^{-1}, \quad (87)$$

$$\Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2 \Gamma_1'^{-1} \Gamma_2^{-1} \Gamma_2'^{-1} = \Gamma_4 \Gamma_3 \Gamma_3'^{-1} \Gamma_4^{-1}, \quad (88)$$

$$\Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1} \Gamma_2'^{-1} = \Gamma_4^{-1} \Gamma_4' \Gamma_4 \Gamma_3 \Gamma_4^{-1} \Gamma_4'^{-1} \Gamma_4, \quad (89)$$

$$\Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1} \Gamma_2'^{-1} = \Gamma_4^{-1} \Gamma_4' \Gamma_4 \Gamma_3 \Gamma_3'^{-1} \Gamma_4^{-1} \Gamma_4'^{-1} \Gamma_4. \quad (90)$$

We also have the following parasitic and projective relations:

$$[\Gamma_1, \Gamma_5] = [\Gamma'_1, \Gamma_5] = [\Gamma_1, \Gamma'_5] = [\Gamma'_1, \Gamma'_5] = e, \quad (91)$$

$$[\Gamma_2, \Gamma_5] = [\Gamma'_2, \Gamma_5] = [\Gamma_2, \Gamma'_5] = [\Gamma'_2, \Gamma'_5] = e, \quad (92)$$

$$\Gamma'_5 \Gamma_5 \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_1 = e. \quad (93)$$

By (87) and (88), we have  $\Gamma_3 = \Gamma'_3$ .

Combining it with (80), (87), and (88), we get

$$\Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_2 \Gamma_1'^{-1} \Gamma_2^{-1} \Gamma_2'^{-1} = \Gamma_3^{-1} \Gamma_4 \Gamma_3. \quad (94)$$

By (84) and (89) we have

$$\Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_2 \Gamma_1'^{-1} \Gamma_2^{-1} \Gamma_2'^{-1} = \Gamma_3^{-1} \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \Gamma_3. \quad (95)$$

It follows that  $\Gamma_4 = \Gamma'_4$ .

Thus, we get the following relations in  $G$ :

$$\Gamma_1 = \Gamma'_1, \quad \Gamma_2 = \Gamma'_2, \quad \Gamma_3 = \Gamma'_3, \quad \Gamma_4 = \Gamma'_4, \quad \Gamma'_5 = \Gamma_4^{-2} \Gamma_5 \Gamma_4^2, \quad (96)$$

$$\Gamma_2 \Gamma_1 \Gamma_2^{-1} = \Gamma_4 \Gamma_3 \Gamma_4^{-1}, \quad (97)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma_4, \Gamma_5 \rangle = e, \quad (98)$$

$$[\Gamma_1, \Gamma_5] = [\Gamma_2, \Gamma_3] = [\Gamma_2, \Gamma_5] = [\Gamma_2^2 \Gamma_1 \Gamma_2^{-2}, \Gamma_4] = [\Gamma_2^2 \Gamma_1 \Gamma_2^{-2}, \Gamma_3^{-2} \Gamma_4 \Gamma_3^2] = e, \quad (99)$$

$$\Gamma_5 \Gamma_4^2 \Gamma_5 \Gamma_3^2 \Gamma_2^2 \Gamma_1^2 = e. \quad (100)$$

In  $G/\langle \Gamma_i^2 \rangle$ ,  $i = 1, 2, 3, 4, 5$ , it is easy to see that the generators are  $\{\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5\}$ , and the relations are the following:

$$\Gamma_1^2 = \Gamma_2^2 = \Gamma_3^2 = \Gamma_4^2 = \Gamma_5^2 = e, \quad (101)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_4 \rangle = \langle \Gamma_4, \Gamma_5 \rangle = e, \quad (102)$$

$$[\Gamma_1, \Gamma_4] = [\Gamma_1, \Gamma_5] = [\Gamma_2, \Gamma_5] = e. \quad (103)$$

It is easy to see that  $G \cong S_5$ , thus  $\pi_1(X_{\text{Gal}})$  is trivial.

This completes the proof. □

### 3.6 A 5-point quintic degeneration

In this subsection we consider a quintic whose degeneration is depicted in Figure 13. This degeneration gives a 5-point, in this case an intersection of five lines and also five planes. According to [16], the configuration in Figure 13 is a Zappatic surface of type  $E_5$ . It is well known that a general del Pezzo  $S$  of degree  $n$  in  $\mathbb{P}^n$  can degenerate to a configuration of points of type  $E_n$ ,  $n = 3, \dots, 9$ . Firstly, we degenerate  $S$  [19] to the cone over a general hyperplane section (elliptic curve) of  $S$ . Secondly, we degenerate the hyperplane section to a cycle of lines.

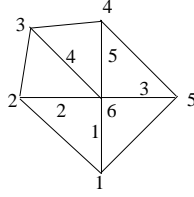


Figure 13: A 5-point quintic degeneration

The regeneration and the related braid monodromy of the 5-points were done in [22]. We use the result from [22, Corollary 2.5] to give the braid monodromy relating to the 5-point.

**Theorem 3.8.**  $\pi_1(X_{Gal})$  of surface with a 5-point quintic degeneration as in Figure 13 is trivial.

*Proof.* The branch curve  $S$  in  $\mathbb{CP}^2$  is an arrangement of 5 lines. We regenerate each vertex in turn and compute group  $G$ .

Vertices 1, 2, 3, 4, and 5 are 1-points, therefore the related braid monodromy factors are  $Z_{11'}$ ,  $Z_{22'}$ ,  $Z_{44'}$ ,  $Z_{55'}$ , and  $Z_{33'}$ , respectively, and hence we have the following relations:

$$\Gamma_1 = \Gamma'_1, \quad \Gamma_2 = \Gamma'_2, \quad \Gamma_3 = \Gamma'_3, \quad \Gamma_4 = \Gamma'_4, \quad \Gamma_5 = \Gamma'_5. \quad (104)$$

According to [22, Corollary 2.5], the braid monodromy corresponding to the 5-point yields the following relations in  $G$ :

$$[\Gamma_3, \Gamma_4] = [\Gamma'_3, \Gamma_4] = e, \quad (105)$$

$$\langle \Gamma'_4, \Gamma_5 \rangle = \langle \Gamma'_4, \Gamma'_5 \rangle = \langle \Gamma'_4, \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \rangle = e, \quad (106)$$

$$\langle \Gamma_2, \Gamma_4 \rangle = \langle \Gamma'_2, \Gamma_4 \rangle = \langle \Gamma_2^{-1} \Gamma'_2 \Gamma_2, \Gamma_4 \rangle = e, \quad (107)$$

$$[\Gamma_4 \Gamma_3 \Gamma_4^{-1}, \Gamma'_5 \Gamma_5 \Gamma'_4 \Gamma_5^{-1} \Gamma'_5^{-1}] = [\Gamma_4 \Gamma'_3 \Gamma_4^{-1}, \Gamma'_5 \Gamma_5 \Gamma'_4 \Gamma_5^{-1} \Gamma'_5^{-1}] = e, \quad (108)$$

$$\Gamma_4 \Gamma'_2 \Gamma_2 \Gamma_4 \Gamma_2^{-1} \Gamma'_2^{-1} \Gamma_4^{-1} = \Gamma'_5 \Gamma_5 \Gamma'_4 \Gamma_5^{-1} \Gamma'_5^{-1}, \quad (109)$$

$$[\Gamma_1, \Gamma_4] = [\Gamma'_1, \Gamma_4] = [\Gamma_1, \Gamma'_5 \Gamma_5 \Gamma'_4 \Gamma_5^{-1} \Gamma'_5^{-1}] = [\Gamma'_1, \Gamma'_5 \Gamma_5 \Gamma'_4 \Gamma_5^{-1} \Gamma'_5^{-1}] = e, \quad (110)$$

$$\langle \Gamma'_1, \Gamma_2 \rangle = \langle \Gamma'_1, \Gamma'_2 \rangle = \langle \Gamma'_1, \Gamma_2^{-1} \Gamma'_2 \Gamma_2 \rangle = e, \quad (111)$$

$$\langle \Gamma_4 \Gamma_3 \Gamma_4^{-1}, \Gamma_5 \rangle = \langle \Gamma_4 \Gamma'_3 \Gamma_4^{-1}, \Gamma_5 \rangle = \langle \Gamma_4 \Gamma_3^{-1} \Gamma'_3 \Gamma_3 \Gamma_4^{-1}, \Gamma_5 \rangle = e, \quad (112)$$

$$\Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1} \Gamma'_2^{-1} = \Gamma_4^{-1} \Gamma_5 \Gamma'_4 \Gamma_3 \Gamma_4^{-1} \Gamma_5^{-1} \Gamma_4, \quad (113)$$

$$\Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma'_2 \Gamma_1^{-1} \Gamma_2^{-1} \Gamma_2'^{-1} = \Gamma_4^{-1} \Gamma_5 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma_3'^{-1} \Gamma_4^{-1} \Gamma_5^{-1} \Gamma_4, \quad (114)$$

$$[\Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_2^{-1} \Gamma_2'^{-1}, \Gamma_4^{-1} \Gamma_5 \Gamma_4] = e, \quad (115)$$

$$[\Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_2^{-1} \Gamma_2'^{-1} \Gamma_3^{-1} \Gamma_3'^{-1}, \Gamma_4^{-1} \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \Gamma_4] = e, \quad (116)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_1, \Gamma'_2 \rangle = \langle \Gamma_1, \Gamma_2^{-1} \Gamma'_2 \Gamma_2 \rangle = e, \quad (117)$$

$$\begin{aligned} \langle \Gamma_4 \Gamma_3 \Gamma_4^{-1}, \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \rangle &= \langle \Gamma_4 \Gamma'_3 \Gamma_4^{-1}, \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \rangle \\ &= \langle \Gamma_4 \Gamma_3^{-1} \Gamma'_3 \Gamma_3 \Gamma_4^{-1}, \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \rangle = e, \end{aligned} \quad (118)$$

$$\Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1} \Gamma_2'^{-1} = \Gamma_4^{-1} \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma_3'^{-1} \Gamma_4^{-1} \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \Gamma_4, \quad (119)$$

$$\Gamma'_2 \Gamma_2 \Gamma_1 \Gamma'_2 \Gamma_1^{-1} \Gamma_2^{-1} \Gamma_2'^{-1} = \Gamma_4^{-1} \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma_3'^{-1} \Gamma_4^{-1} \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \Gamma_4, \quad (120)$$

$$[\Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_2'^{-1}, \Gamma_4^{-1} \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \Gamma_4] = e, \quad (121)$$

$$[\Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma_2'^{-1} \Gamma_3^{-1} \Gamma_3'^{-1}, \Gamma_4^{-1} \Gamma_5^{-1} \Gamma'_5 \Gamma_5 \Gamma_4] = e. \quad (122)$$

We also have the following projective relation:

$$\Gamma'_5 \Gamma_5 \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_1 = e. \quad (123)$$

After we simplify the relations in  $G$ , the generators of  $G$  are  $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$ . The relations are the following:

$$[\Gamma_1, \Gamma_4] = [\Gamma_1, \Gamma_5] = [\Gamma_2, \Gamma_5] = [\Gamma_3, \Gamma_4] = [\Gamma_4, \Gamma_2^2 \Gamma_5^2] = e, \quad (124)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma_4, \Gamma_5 \rangle = e, \quad (125)$$

$$\Gamma_2 \Gamma_1 \Gamma_2^{-1} = \Gamma_3^{-1} \Gamma_4^{-1} \Gamma_5 \Gamma_4 \Gamma_3, \quad (126)$$

$$\Gamma_5^2 \Gamma_4^2 \Gamma_3^2 \Gamma_2^2 \Gamma_1^2 = e. \quad (127)$$

Note that by (126),  $\Gamma_2 = \Gamma_1 \Gamma_3^{-1} \Gamma_4^{-1} \Gamma_5 \Gamma_4 \Gamma_3 \Gamma_1^{-1}$ . We substitute this expression in  $\langle \Gamma_1, \Gamma_2 \rangle = e$  and obtain  $\langle \Gamma_1, \Gamma_3 \rangle = e$ . We substitute this expression in the other relations that include  $\Gamma_2$ , and they become redundant. Now we can say that  $G$  is generated by  $\{\Gamma_1, \Gamma_3, \Gamma_4, \Gamma_5\}$ , and considering the quotient  $G/\langle \Gamma_i^2 \rangle$ , we get

$$[\Gamma_1, \Gamma_4] = [\Gamma_1, \Gamma_5] = [\Gamma_3, \Gamma_4] = e, \quad (128)$$

$$\langle \Gamma_1, \Gamma_3 \rangle = \langle \Gamma_3, \Gamma_5 \rangle = \langle \Gamma_4, \Gamma_5 \rangle = e. \quad (129)$$

Thus  $G/\langle \Gamma_1^2, \Gamma_3^2, \Gamma_4^2, \Gamma_5^2 \rangle \cong S_5$ , and it follows that  $\pi_1(X_{\text{Gal}})$  is trivial.

This completes the proof.  $\square$

### 3.7 A 4-point quintic degeneration

In this subsection we consider a quintic that degenerates to a union of planes as shown in Figure 14. This degeneration gives a special 4-point, in this case an intersection of four edges.

A rational normal scroll  $F$  of degree 5 in  $\mathbb{P}^6$  can degenerate to the cone over a hyperplane section of it, which is a rational normal curve  $C$  of degree 5 in  $\mathbb{P}^5$ . Then  $C$  can be degenerated to a chain of lines which is well known. This yields  $F$  degenerates to 5 planes as in Figure 14.

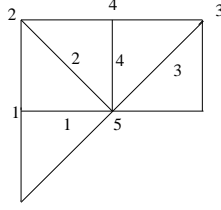


Figure 14: A 4-point quintic degeneration

**Theorem 3.9.**  $\pi_1(X_{Gal})$  of surface with a 4-point quintic degeneration as in Figure 14 is trivial.

*Proof.* The branch curve  $S$  in  $\mathbb{CP}^2$  is an arrangement of 4 lines. We regenerate each vertex in turn and compute group  $G$ .

Vertices 1, 2, 3, and 4 are 1-points, which give rise to the braids  $Z_{1\ 1'}$ ,  $Z_{2\ 2'}$ ,  $Z_{3\ 3'}$ , and  $Z_{4\ 4'}$ , respectively, and hence to the following relations in  $G$ :

$$\Gamma_1 = \Gamma'_1, \quad \Gamma_2 = \Gamma'_2, \quad \Gamma_3 = \Gamma'_3, \quad \Gamma_4 = \Gamma'_4. \quad (130)$$

We have to compute the regeneration of the remaining 4-point (vertex 5) which is not the usual 4-point as in Figure 12. Regenerating it, we get the curve presented in Figure 15.

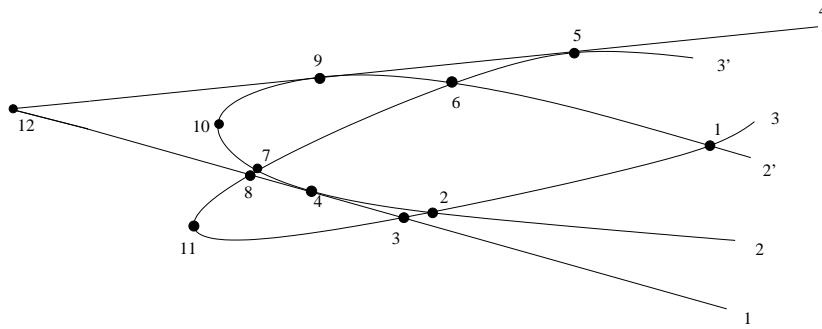


Figure 15: 4-point regeneration

Then the braid monodromy table of the curve in Figure 15 is:

$j$	$x_j$	$\epsilon_{x_j}$	$\delta_{x_j}$
1	$\langle 3, 4 \rangle$	2	$\Delta \langle 3, 4 \rangle$
2	$\langle 2, 3 \rangle$	2	$\Delta \langle 2, 3 \rangle$
3	$\langle 1, 2 \rangle$	2	$\Delta \langle 1, 2 \rangle$
4	$\langle 2, 3 \rangle$	4	$\Delta^2 \langle 2, 3 \rangle$
5	$\langle 5, 6 \rangle$	4	$\Delta^2 \langle 5, 6 \rangle$
6	$\langle 4, 5 \rangle$	2	$\Delta \langle 4, 5 \rangle$

$j$	$x_j$	$\epsilon_{x_j}$	$\delta_{x_j}$
7	$\langle 3, 4 \rangle$	2	$\Delta \langle 3, 4 \rangle$
8	$\langle 2, 3 \rangle$	2	$\Delta \langle 2, 3 \rangle$
9	$\langle 5, 6 \rangle$	4	$\Delta^2 \langle 5, 6 \rangle$
10	$\langle 4, 5 \rangle$	1	$\Delta_{I_2 \mathbb{R}}^{1/2} \langle 4 \rangle$
11	$\langle 1, 2 \rangle$	1	$\Delta_{I_4 I_2}^{1/2} \langle 1 \rangle$
12	$\langle 1, 2 \rangle$		

By Moishezon-Teicher's algorithm [34, 35], we get the braid monodromy factors corresponding to the 4-point (vertex 5):

$$\begin{aligned} \tilde{\Delta}_5 = & Z_{2'3}^2 \cdot Z_{23}^2 \cdot Z_{11',3}^2 \cdot Z_{11',2}^3 \cdot Z_{3',44'}^3 \cdot (\bar{Z}_{2'3'})^{Z_{3',44'}} \cdot (Z_{23'})^{Z_{3',44'}^2 Z_{23}^2 Z_{11',2}^2} \\ & \cdot (Z_{11',3'})^{Z_{3',44'}^2 Z_{11',3}^2 Z_{11',2}^2} \cdot Z_{2',44'}^3 \cdot (Z_{22'})^{Z_{11',2}^2 Z_{2',44'}^2 Z_{3,44'}^{-2}} \cdot (Z_{33'})^{Z_{3',44'}^2} \cdot (Z_{11',44'}^2)^{Z_{11',3}^2 Z_{11',2}^2}. \end{aligned}$$

$\tilde{\Delta}_5$  gives rise to the following relations in  $G$ :

$$[\Gamma_2, \Gamma_3] = [\Gamma'_2, \Gamma_3] = [\Gamma_1, \Gamma_3] = [\Gamma'_1, \Gamma_3] = e, \quad (131)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma'_1, \Gamma_2 \rangle = \langle \Gamma_1^{-1} \Gamma'_1 \Gamma_1, \Gamma_2 \rangle = e, \quad (132)$$

$$\langle \Gamma'_3, \Gamma_4 \rangle = \langle \Gamma'_3, \Gamma'_4 \rangle = \langle \Gamma'_3, \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \rangle = e, \quad (133)$$

$$[\Gamma_3 \Gamma'_2 \Gamma_3^{-1}, \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_4^{-1} \Gamma'_4^{-1}] = [\Gamma_2 \Gamma'_1 \Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma'_1^{-1} \Gamma_2^{-1}, \Gamma_3^{-1} \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_4^{-1} \Gamma'_4^{-1} \Gamma_3] = e, \quad (134)$$

$$[\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma_3^{-1} \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_4^{-1} \Gamma'_4^{-1} \Gamma_3] = e, \quad (135)$$

$$\langle \Gamma_3^{-1} \Gamma'_2 \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_3^{-1} \Gamma'_2 \Gamma_3, \Gamma'_4 \rangle = \langle \Gamma_3^{-1} \Gamma'_2 \Gamma_3, \Gamma_4^{-1} \Gamma'_4 \Gamma_4 \rangle = e, \quad (136)$$

$$\Gamma_2 \Gamma'_1 \Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma'_1^{-1} \Gamma_2^{-1} = \Gamma_3^{-1} \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_2 \Gamma_3^{-1} \Gamma_4^{-1} \Gamma'_4^{-1} \Gamma_3, \quad (137)$$

$$\Gamma_3 = \Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_4^{-1} \Gamma'_4^{-1}, \quad (138)$$

$$[\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma_3^{-1} \Gamma_4 \Gamma_3] = [\Gamma_2 \Gamma'_1 \Gamma_2^{-1}, \Gamma_3^{-1} \Gamma_4 \Gamma_3] = [\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma_3^{-1} \Gamma'_4 \Gamma_3] = [\Gamma_2 \Gamma'_1 \Gamma_2^{-1}, \Gamma_3^{-1} \Gamma'_4 \Gamma_3] = e. \quad (139)$$

We also have the following projective relation:

$$\Gamma'_4 \Gamma_4 \Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_1 = e. \quad (140)$$

Now, we simplify the group  $G$  by using (130), (131), (132) and (133); then we have

$$[\Gamma_1, \Gamma_3] = [\Gamma_2, \Gamma_3] = [\Gamma_2, \Gamma_4^2 \Gamma_3 \Gamma_4^{-2}] = e, \quad (141)$$

$$[\Gamma_1^{-2} \Gamma_2 \Gamma_1^2, \Gamma_4^2 \Gamma_3 \Gamma_4^{-2}] = [\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma_4^2 \Gamma_3 \Gamma_4^{-2}] = [\Gamma_2 \Gamma_1 \Gamma_2^{-1}, \Gamma_4] = e, \quad (142)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = e, \quad (143)$$

$$\Gamma_3 = \Gamma_4^2 \Gamma_3 \Gamma_4^{-2}, \quad (144)$$

$$\Gamma_1^{-2}\Gamma_2\Gamma_1^2 = \Gamma_4^2\Gamma_2\Gamma_4^{-2}, \quad (145)$$

$$\Gamma_4^2\Gamma_3^2\Gamma_2^2\Gamma_1^2 = e. \quad (146)$$

Thus in  $G/\langle\Gamma_i^2\rangle$ , we have the following relations:

$$[\Gamma_1, \Gamma_3] = [\Gamma_2, \Gamma_3] = [\Gamma_2\Gamma_1\Gamma_2, \Gamma_4] = e, \quad (147)$$

$$\langle\Gamma_1, \Gamma_2\rangle = \langle\Gamma_2, \Gamma_4\rangle = \langle\Gamma_3, \Gamma_4\rangle = e. \quad (148)$$

It is easy to see that

$$[\Gamma_2\Gamma_1\Gamma_2, \Gamma_3] = \langle\Gamma_2\Gamma_1\Gamma_2, \Gamma_2\rangle = e. \quad (149)$$

We can then choose  $\{\Gamma_2\Gamma_1\Gamma_2, \Gamma_2, \Gamma_3, \Gamma_4\}$  to be the generators of  $G/\langle\Gamma_i^2\rangle$ . It follows that  $G/\langle\Gamma_i^2\rangle \cong S_5$ . Thus  $\pi_1(X_{\text{Gal}})$  is trivial.

This completes the proof.  $\square$

## 4 On a question of Liedtke

In order to compute the fundamental group of the affine piece of Galois cover of surfaces, we introduce the following notions:  $S^{\text{Aff}}$  is the affine part of  $S$  in  $\mathbb{C}^2$ ,  $x_0$  is a point in  $\mathbb{C}^2 - S^{\text{Aff}}$ , and  $\pi_1(\mathbb{C}^2 - S^{\text{Aff}}, x_0)$  is the fundamental group of the complement of  $S^{\text{Aff}}$  in  $\mathbb{C}^2$ . This group is in fact  $G$ , without the projective relation.

We have the following exact sequence ([29, Section 3], [33, Chapter 0.3]):

$$1 \rightarrow \pi_1(X_{\text{Gal}}^{\text{Aff}}) \rightarrow \frac{\pi_1(\mathbb{C}^2 - S^{\text{Aff}}, x_0)}{\langle\Gamma_i^2\rangle} \xrightarrow{\psi} S_n \rightarrow 1. \quad (150)$$

Liedtke introduced a normal subgroup  $C^{\text{Aff}}$  of  $\pi_1(\mathbb{C}^2 - S^{\text{Aff}}, x_0)$ :

**Definition 4.1.** [29, Definition 3.3]  $C^{\text{Aff}}$  is the subgroup normally generated by the following elements inside  $\pi_1(\mathbb{C}^2 - S^{\text{Aff}}, x_0)$ :

$[\gamma\Gamma_i\gamma, \Gamma_j^{-1}]$ , if  $\psi(\gamma\Gamma_i\gamma^{-1})$  and  $\psi(\Gamma_j)$  are disjoint transpositions,

$\langle\gamma\Gamma_i\gamma, \Gamma_j^{-1}\rangle$ , if  $\psi(\gamma\Gamma_i\gamma^{-1})$  and  $\psi(\Gamma_j)$  have precisely one letter in common,

where  $\gamma$  runs through  $\pi_1(\mathbb{C}^2 - S^{\text{Aff}}, x_0)$ .

$C^{\text{Aff}}$  is trivial if the commutator and triple commutator relations hold in the above definition. After taking the quotient of the short exact sequence (150) by  $C^{\text{Aff}}$ , the resulting short exact sequence splits in a very nice way. Then he simplifies the computation of  $\pi_1(\mathbb{C}^2 - S^{\text{Aff}}, x_0)$  and  $\pi_1(\mathbb{CP}^2 - S)$  (cf: [29]). He then states the following question, which is the main unsolved problem:

**Question 4.1.** Is  $C^{\text{Aff}}$  trivial for every generic projection of degree  $n \geq 5$ ?

This is true in all known examples (such as Hirzebruch surface,  $\mathbb{CP}^1 \times \mathbb{CP}^1$ ,  $\mathbb{CP}^1 \times C_g, \dots$ ). Here, we give a counter-example of degree 5.



**Theorem 4.2.**  $C^{\text{Aff}}$  of surfaces with the degeneration as in Figure 10 is non-trivial.

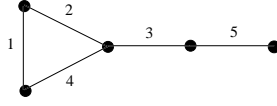
*Proof.* In the proof of Theorem 3.6, we simplify the relations without using the projective relation, thus  $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$  are generators of  $\pi_1(\mathbb{C}^2 - S^{\text{Aff}}, x_0)/\langle \Gamma_i^2 \rangle$ , and we have the following relations:

$$[\Gamma_1, \Gamma_3] = [\Gamma_1, \Gamma_5] = [\Gamma_2, \Gamma_5] = [\Gamma_1, \Gamma_2 \Gamma_4 \Gamma_2] = [\Gamma_4, \Gamma_5] = e,$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma_2, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_4 \rangle = \langle \Gamma_3, \Gamma_5 \rangle = e.$$

The projection  $\psi : \frac{\pi_1(\mathbb{C}^2 - S^{\text{Aff}}, x_0)}{\langle \Gamma_i^2 \rangle} \rightarrow S_n$  is defined as follows:

$$\Gamma_1 \mapsto (12), \Gamma_2 \mapsto (13), \Gamma_3 \mapsto (34), \Gamma_4 \mapsto (23), \Gamma_5 \mapsto (45).$$



It is easy to see that  $\psi(\Gamma_3 \Gamma_4 \Gamma_3^{-1})$  and  $\psi(\Gamma_2)$  are disjoint transpositions. By [39, pp.141],  $[\Gamma_3 \Gamma_4 \Gamma_3, \Gamma_2^{-1}]$  is a generator of  $\text{Ker} \psi$  which is non trivial. Thus,  $C^{\text{Aff}}$  is non-trivial.

This completes the proof.  $\square$

## 5 Appendix: A missing case in [9]

In [9], the authors classified fundamental groups of Galois covers of surfaces of degree 4 degenerating to nice plane arrangements, but did not specifically include the following case (Figure 16). Indeed, the rational normal scroll  $F$  degenerates to a union of planes in Figure 16 (this is a toric degeneration, corresponding to the subdivision of the  $(1, 2)$ -rectangle).

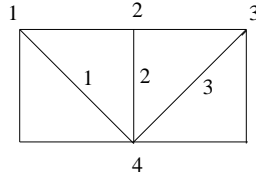


Figure 16: Degree 4

**Theorem 5.1.**  $\pi_1(X_{\text{Gal}})$  of surfaces with the degeneration as in Figure 16 is trivial.

*Proof.* Vertices 1, 2, and 3 are 1-points, therefore the related braids are  $Z_{11'}$ ,  $Z_{22'}$ , and  $Z_{33'}$  respectively, and hence the relations in  $G$  are the following:

$$\Gamma_1 = \Gamma'_1, \quad \Gamma_2 = \Gamma'_2, \quad \Gamma_3 = \Gamma'_3. \quad (151)$$

Vertex 4 is a 3-point (line 1 and 3 are diagonals) that regenerates to a line 2 tangent to two conics,  $(1, 1')$  and  $(3, 3')$ .

We give the following relations in  $G$ :

$$[\Gamma_1, \Gamma_3] = [\Gamma'_1, \Gamma_3] = [\Gamma_1, \Gamma_3^{-1} \Gamma'_3 \Gamma_3] = [\Gamma'_1, \Gamma_3^{-1} \Gamma'_3 \Gamma_3] = e, \quad (152)$$

$$\langle \Gamma_2, \Gamma_3 \rangle = \langle \Gamma'_2, \Gamma_3 \rangle = \langle \Gamma_2^{-1} \Gamma'_2 \Gamma_2, \Gamma_3 \rangle = e, \quad (153)$$

$$\Gamma'_3 = \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma_3 \Gamma_2^{-1} \Gamma'_2 \Gamma_3^{-1}, \quad (154)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_1, \Gamma'_2 \rangle = \langle \Gamma_1, \Gamma_2^{-1} \Gamma'_2 \Gamma_2 \rangle = e, \quad (155)$$

$$\Gamma'_1 = \Gamma_1 \Gamma'_2 \Gamma_2 \Gamma_1 \Gamma_2^{-1} \Gamma'_2 \Gamma_1^{-1}. \quad (156)$$

$$\Gamma'_3 \Gamma_3 \Gamma'_2 \Gamma_2 \Gamma'_1 \Gamma_1 = e. \quad (157)$$

After an easy simplification, we get the following relations (combining projective relation):

$$[\Gamma_1, \Gamma_3] = e, \quad (158)$$

$$\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_3 \rangle = e, \quad (159)$$

$$\Gamma_3 = \Gamma_2^{-2} \Gamma_3 \Gamma_2^2, \quad \Gamma_1 = \Gamma_2^{-2} \Gamma_1 \Gamma_2^2, \quad (160)$$

$$\Gamma_3^2 \Gamma_2^2 \Gamma_1^2 = e. \quad (161)$$

Thus  $G/\langle \Gamma_1^2, \Gamma_2^2, \Gamma_3^2 \rangle \cong S_4$ , hence  $\pi_1(X_{\text{Gal}})$  is trivial.

This completes the proof.  $\square$

## References

- [1] Amram, M., Ciliberto, C., Miranda, R., Teicher, M., *Braid monodromy factorization for a non-prime K3 surface branch curve*, Israel Journal of Mathematics, **170**, (2009), 61–93.
- [2] Amram, M., Cohen, M., Teicher, M., *Links arising from braid monodromy factorizations*, Journal of Knot Theory and Its Ramifications, **23**, no. 2, (2014), (32 pages).
- [3] Amram, M., Dettweiler, M., Friedman, M., Teicher, M., *Fundamental group of the complements of the Cayley's singularities*, Beiträge zur Algebra und Geometrie, Contributions to Algebra and Geometry, **50**, no. 2, (2009), 469–482.
- [4] Amram, M., Friedman, M., Teicher, M., *The fundamental group of the complement of the branch curve of the second Hirzebruch surface*, Topology, **48**, (2009), 23–40.
- [5] —, *The fundamental group of the branch curve of the complement of the surface  $\mathbb{CP}^1 \times T$* , Acta Mathematica Sinica, **25**(9), (2009), 1443–1458.
- [6] Amram, M., Goldberg, D., Teicher, M., Vishne, U., *The fundamental group of a Galois cover of  $\mathbb{CP}^1 \times T$* , Algebraic & Geometric Topology, **2**, (2002), 403–432.

- [7] Amram, M., Goldberg, D., *Higher degree Galois covers of  $\mathbb{CP}^1 \times T$* , Algebraic & Geometric Topology, **4**, (2004), 841–859.
- [8] Amram, M., Garber, D., Shwartz, R., Teicher, M., *8-point – regenerations and applications*, Advances in Geometric Analysis, Advanced Lectures in Mathematics (ALM)-series, International Press of Boston, in cooperation with the Higher Education Press (Beijing) and the Stefan Banach International Mathematical Centre, ALM 21, (2012), 307–342.
- [9] Amram, M., Lehman, R., Shwartz, R., Teicher, M., *Classification of fundamental groups of Galois covers of surfaces of small degree degenerating to nice plane arrangements*, in Topology of algebraic varieties and singularities, volume 538, Contemp. Math., 63–92, Amer. Math. Soc., Providence, RI, 2011.
- [10] Amram, M., Ogata, S., *Toric varieties–degenerations and fundamental groups*, Michigan Math. J., **54**, (2006), 587–610.
- [11] Amram, M., Teicher, M., *The fundamental group of the complement of the branch curve of  $T \times T$  in  $\mathbb{C}^2$* , Osaka Journal of Math., **40**, (2003), 1–37.
- [12] —, *On the degeneration, regeneration and braid monodromy of  $T \times T$* , Acta Appl. Math., **75**, (2003), 195–270.
- [13] Amram, M., Teicher, M., Vishne, U., *The fundamental group of the Galois cover of the surface  $T \times T$* , International J. of Algebra and Computation, **18**, no. 8, (2008), 1259–1282.
- [14] Artin, E., *Theory of braids*, Ann. Math., **48**, (1947), 101–126.
- [15] Auroux, D., Donaldson, S., Katzarkov, L., Yotov, M., *Fundamental groups of complements of plane curves and symplectic invariants*, Topology, **43**, (2004), 1285–1318.
- [16] Calabri, A., Ciliberto, C., Flamini, F., Miranda, R., *On the  $K^2$  of degenerations of surfaces and the multiple point formula*, Ann. Math., **165**, (2007), 335–395.
- [17] Catanese, F., *On the moduli spaces of surfaces of general type*, J. Differential Geometry, **19**, (1984), 483–515.
- [18] —, *(Some) old and new results on algebraic surfaces*, First European Congress of Mathematics, Birkhauser Basel, (1994), 445–490.
- [19] Ciliberto, C., Lopez, A., Miranda, R., *Projective degenerations of K3 surfaces, Gaussian maps, and Fano threefolds*, Invent. Math., **114**, (1993), 641–667.
- [20] Cayley, A., *A Memoir on Cubic Surfaces*, Philos. Trans. Roy. Soc. London, **159**, (1869), 231–326.
- [21] Chisini, O., *Sulla identità birazionale delle funzioni algebriche di due variabili dotate di una medesima curva di diramazione*, Rend. Istit. Lombardo, **77**, (1944), 339–356.

- [22] Friedman, M., Teicher, M., *The regeneration of a 5-point*, Pure and Applied Math. Quarterly, **4**, no. 2, (2008), 383–425.
- [23] —, *On fundamental groups related to the Hirzebruch surface  $F_1$* , Sci. China Ser. A., **51**, (2008), 728–745.
- [24] —, *On fundamental groups related degenerate surfaces: conjectures and examples*, Ann. Sc. Norm. Super. Pisa Cl. Sci., Vol. XI(5), (2012), 565–603.
- [25] Gieseker, D., *Global moduli for surfaces of general type*, Invent. Math., **41**, (1977), 233–282.
- [26] Goncalves, D. L., Guaschi, J., *The braid groups of the projective plane*, Algebraic & Geometric Topology, **4**, (2004), 757–780.
- [27] Kulikov, V., *On Chisini’s conjecture*, Izv. Math., **63**, no. 6, (1999), 1139–1170.
- [28] —, *On Chisini’s conjecture, II*, Izv. Math., **72**, no. 5, (2008), 901–913.
- [29] Liedtke, C., *Fundamental groups of Galois closures of generic projections*, Trans. of AMS, **362**, (2010), 2167–2188.
- [30] Manetti, M., *On the Chern numbers of surfaces of general type*, Composito Mathematica, **92**, (1994), 285–297.
- [31] Moishezon, B., Robb, A., Teicher, M., *On Galois covers of Hirzebruch surfaces*, Math. Ann., **305**, (1996), 493–539.
- [32] Moishezon, B., Teicher, M., *Galois covers in theory of algebraic surfaces*, Proceedings of Symposia in Pure Math., **46**, (1987), 47–65.
- [33] —, *Simply connected algebraic surfaces of positive index*, Invent. Math., **89**, (1987), 601–643.
- [34] —, *Braid group technique in complex geometry I, Line arrangements in  $\mathbb{CP}^2$* , Contemporary Math., **78**, (1988), 425–555.
- [35] —, *Braid group technique in complex geometry II, From arrangements of lines and conics to cuspidal curves*, Algebraic Geometry, Lect. Notes in Math., **1479**, (1991), 131–180.
- [36] —, *Braid group technique in complex geometry III: Projective degeneration of  $V_3$* , Contemp. Math., **162**, (1994), 313–332.
- [37] —, *Braid group technique in complex geometry IV: Braid monodromy of the branch curve  $S_3$  of  $V_3 \rightarrow \mathbb{CP}^2$  and application to  $\pi_1(\mathbb{C}^2 - S_3, *)$* , Contemporary Math., **162**, (1994), 332–358.
- [38] —, *Braid group technique in complex geometry V: The fundamental group of a complement of a branch curve of a Veronese generic projection*, Comm. Anal. Geom., **4**, (1996), 1–120.
- [39] Rowen, L., Teicher, M., Vishne, U., *Coxeter covers of the symmetric groups*, J. Group Theory, **8**, (2005), 139–169.

- [40] Salmon, G., *A treatise on analytic geometry of three dimensions*, Fifth edition, revised by R. A. P. Rogers, Fellow of Trinity College, Dublin. Volume 1. London, Longmans, Green and Company, 1912.
- [41] Schläfli, L., *On the distribution of surfaces of third order into species, in reference to the absence or presence of singular points, and the reality of their lines*, Philos. Trans. Roy. Soc. London, **153**, (1863), 193–241.
- [42] Teicher, M., *New invariants for surfaces*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), 271–281, Contemp. Math., **231**, Amer. Math. Soc., Providence, RI, 1999.
- [43] van Kampen, E. R., *On the fundamental group of an algebraic curve*, Amer. J. Math., **55**, (1933), 255–260.
- [44] Zariski, O., *On the topological discriminant group of a Riemann surface of genus  $p$* , Amer. J. Math., **59**, (1937), 335–358.